4H Project : Curves and Singularities

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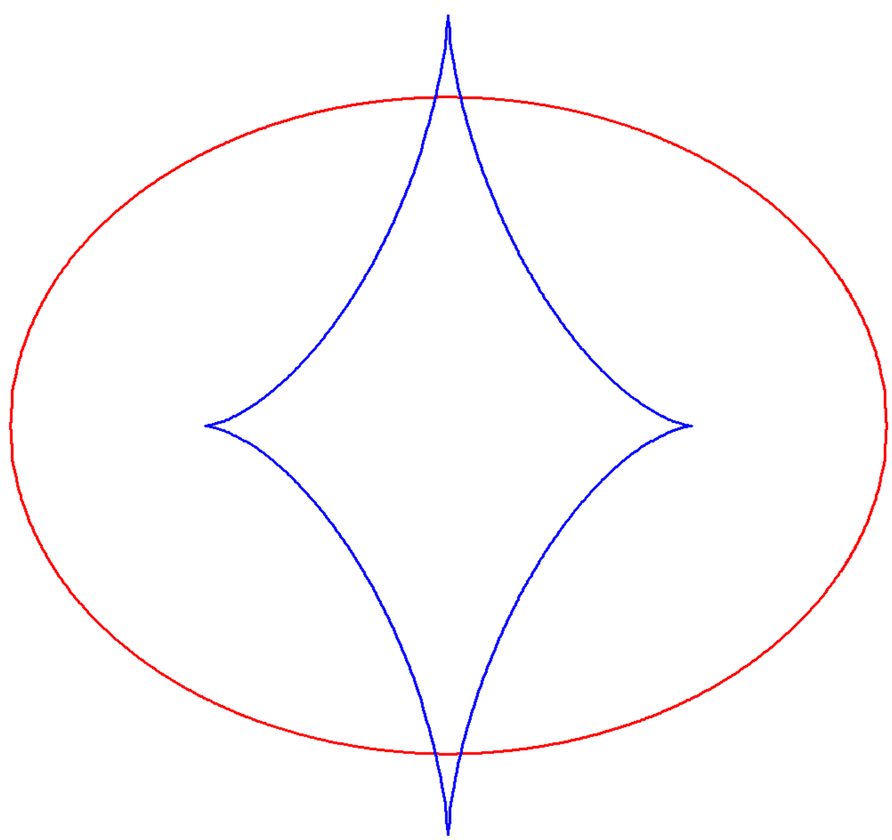

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1 Introduction

This project looks at the geometry of curves in two, three and higher dimensions, studied from the point of view of singularity theory. The central idea is that the singularities of functions defined on curves can be used to reveal interesting geometric information about the curve, or about related curves. In particular, we demonstrate the links between the singularities of the distance-squared function (and its affine variants) with points where the curvature has an extremum, and also with the local structure of the evolute.

In Chapter 2 (*Curves and functions on them*), we introduce Euclidean plane curves, and outline the basic notions associated with a curve. The important concept of the curvature of a curve is introduced, as well the distance-squared function. These ideas are the extended to cover space curves and curves in higher dimensions. Chapter 3 (*Technical tools*) outlines some of the key concepts form function theory that are used. These include 'right equivalence', which allows us to replace a function defining a curve at a point with a much simpler function. In addition, the conditions to treat a family of functions as one single function (an 'unfolding') are given

Chapter 4 (*Singularities*) provides the core of this work. We see in detail how the singularities distance-squared function correspond to vertices of the curve (points where the curvature has an extremum), and how the type of singularity tells us the type of vertex. This is done separately for plane, space and higher dimensional curves. For plane curves, the local structure of the evolute can also be determined from this information.

Chapter 5 (*Generic properties*) looks at what properties are true for 'almost all' curves. A proper definition of 'almost all' is given, and such properties are known as 'generic' properties. The Monge-Taylor map and transversality theorem are used to establish certain properties as being generic.

After three Chapters on Euclidean curves, Chapter 6 (*Affine curves*) generalises the results. The concept of an affine transformation is introduced, and compared with a Euclidean transformation. As Euclidean curvature is not preserved under affine transformations, the notion of affine curvature is introduced, along with affine versions of vertices, generalised curvature, frame vectors and the distance-squared function met in Chapter 2. The techniques from Chapter 4 are used to show that despite the differences between the affine and Euclidean versions of these concepts, the results obtained are surprisingly similar.

Finally, in Chapter 7, we look at possible extensions of the material covered, both into new and covered areas. In particular, a possible reason for the similarity of affine and Euclidean results is discussed. A glossary of the many Greek and Roman letters used in this work is given at the end.

1.1 Sources

The main source for this work has been the book by Bruce & Giblin [2]. This provided the material on singularities and how to use them to establish geometric properties of curves, which forms the bulk of this work. Material from the University's Differential Geometry III course was used in Chapter 2, and is assumed to be prior knowledge. The section on generalised curvatures comes from the web article [10]. Although the results of Chapter 4 relating to singularities and the bifurcation set of the distance squared function in two and three dimensions are from Bruce & Giblin [2], the extension into n dimensions is entirely my own work.

Chapter 6 is based on two papers by Izumiya & Sano covering plane curves [7] and space curves [8], and the forthcoming paper on higher dimensional curves by Davis [5]. However, the proposition proved in section 6.4 is original as far as I can tell. The books by Arnold [1], McCleary [9], Cartan [4] and Brücker & Lander [3] were used for background reading, while the paper by Fidal [6] is referenced due to the use of a specific formula from it in section 6.3.

Diagrams were produced with the aid of the computer programmes 'Autograph' [11], 'Excel' [12], 'Paintbrush' [13] and 'Paint Shop Pro' [14].

Title page The graphic on the title page depicts an ellipse, along with its evolute, an astroid. Using the information in this project, the four cusps of the astroid imply that the ellipse has four vertices.

2 Curves and functions on them

This chapter gives a basic overview of Euclidean curves in two and more dimensions. A formal definition of a plane curve is given, and various associated notions such as speed, k-point contact and tangent and normal vectors are introduced. This leads onto to an introduction to the fundamental property of the 'curvature' of a curve. The curvature (and points where its derivative vanishes) are of central importance to this work. Two useful functions, the height function and the distance-squared function are defined, and the latter will be shown in this work to have close associations with the derivative of the curvature. The final two sections generalise these ideas to three and n dimensions respectively. In particular, the single curvature function from two dimensions is extended to $n-1$ 'curvature functions'.

2.1 Plane curves

A plane curve γ is defined to be a smooth, single-valued function from the real numbers (or some real interval) to the standard Euclidean plane, $\mathbb{R}^2$. Here, 'smooth' means that all derivatives of γ exist and are continuous. The function is generally written in the form $\gamma(t) = ((x(t), y(t))$.

A curve can be thought of as the position of some point particle as a function of time. As time evolves, the particle 'traces out' the curve. This naturally leads to defining the 'speed' of the curve as the scalar $\gamma'(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$. A curve is said to be unit speed if $\dot{\gamma}(t) = 1\ \forall t$; and it is said to be regular if $\dot{\gamma}(t) \neq 0\ \forall t$, where $\dot{\gamma}(t) = \frac{d\gamma}{dt}$. All curves in this document are assumed to be regular unless stated otherwise. The notion of speed leads onto the notion of 'velocity', and this is defined as the the vector $\mathbf{V}(t) = (\dot{x}, \dot{y})$. This vector is always tangent to the curve, and gives rise to the more useful unit tangent vector, $\mathbf{T}(t) = \frac{(\dot{x},\dot{y})}{\sqrt{\dot{x}^2+\dot{y}^2}}$. (Note that the velocity vector is equal to the speed times the unit tangent vector). Any tangent vector on a plane defines two possible normal vectors, and the convention is to use the left-handed unit normal vector, $\mathbf{N}(t) = \frac{(-\dot{y},\dot{x})}{\sqrt{\dot{x}^2+\dot{y}^2}}$

Occasionally it is simpler to use the 'level set' representation of a curve. Given some single valued function $C(x, y) : I \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^2$, then this defines a curve given by $\{(x, y) \in \mathbb{R}^2 : C(x, y) = 0\}$. While the two definitions produce equivalent curves, changing from one to the other can be difficult in practice.

2.2 Intersection and k-point contact

When dealing with more than one curve in a plane, one question that arises is whether two curves intersect or not. If they intersect, then it also useful to determine the exact nature of that intersection - for example, whether the curves are tangent to each other.

Given a plane curve $\gamma(t) = ((x(t), y(t))$, and some other curve in $\mathbb{R}^2$ defined by $C(x, y) = 0$, then the intersection of γ and C is, by definition, $\gamma \cap C = \{\gamma(t) \mid C(x(t), y(t)) = 0\}$. If we then define $f(t) \overset{def}{=} C(\gamma(t)) = C((x(t), y(t))$, then the intersection is simply given by

$$\gamma \cap C = \{\gamma(t) \mid f(t) = 0\}$$

Having dealt with dealt with the existence of intersection points, logically the next step is to determine how many intersection points there are. Although this is a fascinating study in its own right, it is not dealt with here. However, this brings us to the classification of intersection points. For this we introduce the notion of k-point contact. Consider a point $\gamma(t_0)$ where $f(t_0) = 0$ (f being as above). Then if $f^{(i)}(t_0) = 0$ for $i = 1$ to $k - 1$, and $f^{(k)}(t_0) \neq 0$ then γ is said to have 'k-point contact' with C at $t = t_0$. Thus if two curves have 0-point contact, they intersect but are not tangent to one another, because their derivatives are not equal. If two curves have at least 1-point contact, then they are tangent. A curve is said to have an ordinary (respectively higher) inflexion at a point if the tangent line has 3-point (respectively at least 4-point) contact with the curve.

2.3 Reparameterisation

It is often useful to have a way of changing a curve $\gamma(t)$ into a unit speed version, such the curve is geometrically the same. This can be done by reparameterising the curve.

Let $\gamma(t) : t \in J \subseteq \mathbb{R} \mapsto \mathbb{R}^2$ be a regular curve. Then to reparameterise the curve we replace the parameter t with a smooth function $h(t) : t \in J \subseteq \mathbb{R} \mapsto I \subseteq \mathbb{R}$, subject to the conditions $h'(t) \neq 0\ \forall t \in J$; $h(J) = I$. To make a curve unit speed, simply use $h(t) = s^{-1}(t)$, where $s^{-1}(t)$ is the inverse of the arc-length function, given in the next section.

2.4 Functions on plane curves

In this section, various functions on curves are defined. The motivation is that such functions can be analysed algebraically to reveal interesting geometric information about a curve. This is done by finding values for which the function has particular properties, such as certain derivatives vanishing. These values then correspond to points on the curve with the desired property.

Arc length It is often useful to know the length along a curve between two points. If we return to our visualisation of a curve being the path traced out by a point particle over time, then the arc length is simply the distance traveled by that point. If we write $\gamma(t) = (x(t), y(t))$, then the arc length $s(t)$ is given by

$$s(t) = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

If the curve is unit speed, then the integrand is identically equal to one, and the integral simplifies to yield $s(t) = t_2 - t_1$.

Curvature One fundamental characteristic of a curve is its curvature, $\kappa(t)$, and there are several ways this can be defined. The simplest is to consider a circle having (at least) 3-point contact with γ at $\gamma(t)$. That circle has a radius, $R(t)$ (see Figure 1 on page 9). The curvature is then given by

$$\kappa(t) = \frac{1}{R(t)} = \frac{\dot{x}\ddot{y} - \dot{x}\ddot{y}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

The derivation of the second formula in the above equation is somewhat complex, and will not be reproduced here (it is done as part of the University's Differential Geometry III course). The term in the denominator is equal to the cube of the speed of the curve, and is thus equal to one for a unit speed curve. It follows from the definition that a circle has constant curvature. The centre of the circle is known as the 'centre of curvature' and lies on the normal line to the curve. The locus of the centre of curvature is called the 'evolute', and is given by $\xi(t) = \gamma(t) + R(t)\mathbf{N}(t) = \gamma(t) + \frac{\mathbf{N}(t)}{\kappa(t)}$

If the curve is a straight line, then there is no Euclidean circle with three point contact. In this case, the curvature is defined to be zero. (In Riemannian geometry, a straight line is just a circle of infinite radius, and this

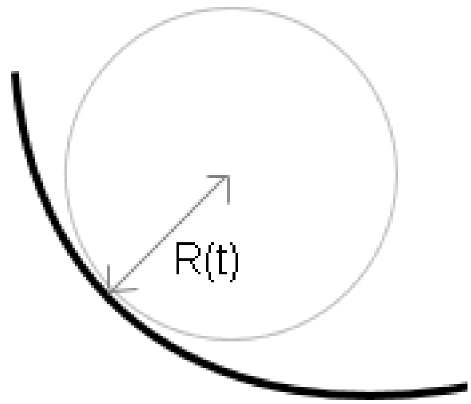


Figure 1: Radius of curvature

implies the same result). A point where the curvature has a local maximum or minimum is known as a 'vertex'. When $\kappa'' \neq 0$, the point is called an 'ordinary vertex'; when $\kappa'' = 0$, the point is called a 'higher vertex'.

An alternative definition of the curvature is as follows. Given a point P at $\gamma(t_0)$, the tangent at P either intersects the x-axis with angle θ, or is parallel to the x-axis. The latter case is taken as $\theta = 0$. This angle is given by $\tan\theta = \frac{\dot{y}}{\dot{x}}$. If the arc-length of γ is denoted by s(t), then the curvature is $\kappa = \frac{d\theta}{ds}$ i.e. the rate of change of tangent angle with respect to arc length. (Note that if γ is unit speed, then $s(t) = \gamma(t)$). Assuming γ is unit speed, than we have:

$$\frac{d\theta}{ds} = \frac{d\theta}{d\gamma}$$

$$= \frac{d\theta}{dt}.\frac{dt}{d\gamma}$$

$$= \frac{d\theta}{dt}.\frac{1}{\frac{d\gamma}{dt}}$$

as γ is unit speed

$$= \frac{d\theta}{dt}$$

$$= \frac{d}{dt}(\arctan(\frac{\dot{y}}{\dot{x}}))$$

$$
\begin{aligned}
&= \frac{\dot{x}^2}{\dot{x}^2+\dot{y}^2}.\frac{\ddot{y}\dot{x}-\dot{y}\ddot{x}}{\dot{x}^2} \\
&= \frac{\ddot{y}\dot{x}-\dot{y}\ddot{x}}{\dot{x}^2+\dot{y}^2}
\end{aligned}
$$

as γ is unit speed, $\dot{x}^2+\dot{y}^2=1$

$$
\begin{aligned}
&= \ddot{y}\dot{x}-\dot{y}\ddot{x} \\
&= \kappa(t)
\end{aligned}
$$

A much simpler, though less intuitive way of defining the curvature of a plane curve is via the Serret-Frenet formulae. These relate the tangent and normal vectors with their derivatives. For a unit speed curve, they state that

$$
\begin{aligned}
\mathbf{T}' &= \kappa\mathbf{N} \\
\mathbf{N}' &= -\kappa\mathbf{T}
\end{aligned}
$$

We will see later how these formulae extend for curves in higher dimensions.

Distance-squared function The distance-squared function, $D_{\mathbf{u}}(t)$ gives the square of the distance from a point $\mathbf{u} \in \mathbb{R}^2$ to γt (see Figure 2 on 11), and is given by

$$D_{\mathbf{u}}(t) = |\ \gamma(t) - \mathbf{u}\ |^2 = (\gamma(t)-\mathbf{u}).(\gamma(t)-\mathbf{u})$$

The distance squared function can be used to reveal useful information about the curvature of a plane curve. We will see later that considering the points at which the derivatives of the function vanish leads directly to points where the curve has a vertex. Further, these points correspond to cusps on the evolute.

Height function The height function gives the shortest distance from a point on a curve to a line (see Figure 3 on page 11). However, as only the derivatives of this function are ever considered, it is only ever defined up to an additive function. (In other words, it becomes a function of only the gradient of the line). Given a line perpendicular to a vector $\mathbf{u}$, then the perpendicular distance from that line (to within an additive constant) is given by

$$H_{\mathbf{u}}(t) = \gamma(t).\mathbf{u}$$

Although we do not show this explicitly, the height function can be used to find points where there are inflexions on a curve.

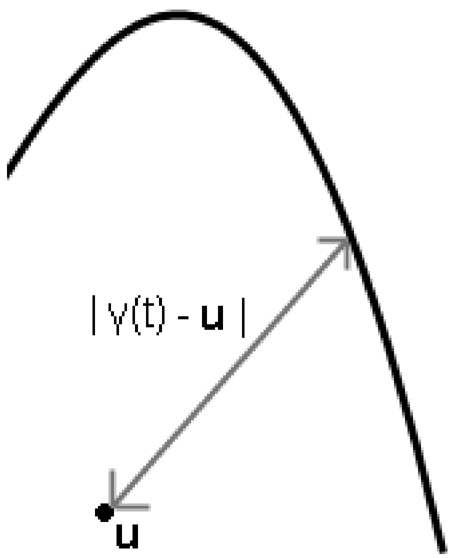


Figure 2: Distance-squared function

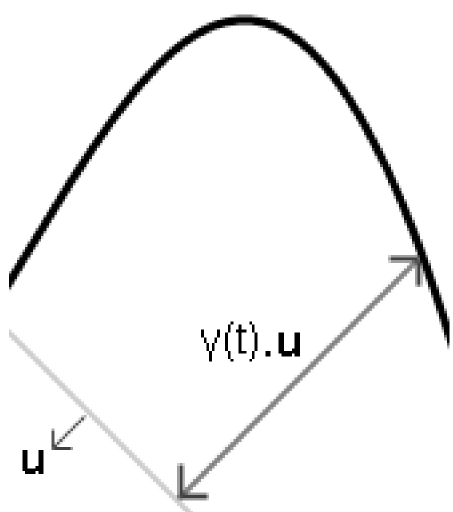


Figure 3: Height function

2.5 Space curves

Up to this point we have only considered curves in the plane. However, the ideas for plane curves extend naturally from two to three dimensions. We define a space curve α as a smooth, single-valued function $\alpha : I \subset \mathbb{R} \mapsto \mathbb{R}^3$. Writing $\alpha(t) = (x(t), y(t), z(t))$, then we can define the speed of the curve α to be $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$, with the definitions of 'regular' and 'unit speed' as for plane curves. We shall assume that all our space curves are unit speed from now on (this assumption involves no loss of generality - see previous section on reparameterisation). To reflect this, we shall use s as the parameter, as unit speed curves are parameterised by their arc length.

The definition of the tangent vector also extends simply from two dimensions to three and is given by

$$\mathbf{T}(s) = \frac{(x'(s), y'(s), z'(s))}{\sqrt{x'(\dot{s})^2 + y'(\dot{s})^2 + z'(\dot{s})^2}}$$

The line containing the tangent vector at a point $\alpha(s_0)$ is also the line having highest contact with the curve at that point. (This is also true of planar curves). However, for the normal vector, three dimensions results in a choice of infinitely many vectors perpendicular to the tangent vector. To resolve this, we consider the plane having highest contact with the curve at $\alpha(s_0)$. It can be shown that this plane always contains the tangent line, and thus two possible normal vectors. As for two dimensions, we pick the left handed one, and this becomes our principal normal vector, $\mathbf{N}(s)$. This definition fails if $\mathbf{T}'(s) = 0$, as the plane with highest point contact is not then unique. However, this is only the case if the curvature (defined below) vanishes - and then geometrically, no normal is any more important than another. We will assume that all space curves have non-zero curvature from now on.

To find the curvature of a space curve at a point on a curve, consider a sphere having (at least) 4-point contact with the curve at that point. Then this sphere has a radius, $1/\kappa$, where κ is the curvature at that point. If the curve is merely a straight line, then the curvature is defined to be zero. The curvature links in with the normal vector and the derivative of the tangent vector via one of the 3D Serret-Frenet formulae:

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$$

Alternatively, the normal vector can be algebraically defined simply as

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{|\mathbf{T}'(s)|}$$

The two definition together imply that $\kappa(s) = |\mathbf{T}'(\mathbf{s})|$. Given the normal and tangent vectors, we can then define the binormal vector via $\mathbf{B} = \mathbf{T} \text{ x } \mathbf{N}$. Clearly, this is perpendicular to both the normal and tangent vectors. The plane spanned by the normal and and binormal vectors is called the normal plane; the plane spanned by the tangent and normal vectors is called the osculating plane.

The three vectors and their derivatives are linked by the remaining two Serret-Frenet formulae:

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$$

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$$

The function $\tau(s)$ is called the 'torsion' of the curve, and can be thought of as a measure of how much the curve 'curves out' of the osculating plane. In fact, the torsion is identically zero if and only if the curve is planar.

The three formulae can be written in matrix form, which allow them to be more easily remembered, and also generalised into higher dimensions:

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

Example Consider the curve given $\beta(s) = (r\sin(s), r\cos(s), cs)$, which is a helix with radius r and pitch c. This curve has speed $|\beta'| = r^2 + c^2$, and the unit speed version of this curve is therefore:

$$\tilde{\beta}(\tilde{s}) = \left(r\sin(\frac{\tilde{s}}{r^2+c^2}), r\cos(\frac{\tilde{s}}{r^2+c^2}), c\frac{\tilde{s}}{r^2+c^2}\right)$$

However, we shall use β rather than $\tilde{\beta}$ in this example for brevity. The curvature, torsion and tangent, normal and binormal vectors, are given by:

$$\mathbf{T}(s) \quad = \quad \left(\frac{-r\sin(s)}{r^2+c^2}\ , \ \frac{-r\cos(s)}{r^2+c^2}\ , \ \frac{c}{r^2+c^2}\right)$$

$$\begin{aligned}
\mathbf{N}(s) &= (-\cos(s)\ ,\ -\sin(s)\ ,\ 0) \\
\mathbf{B}(s) &= \left(\frac{c\sin(s)}{r^2+c^2}\ ,\ \frac{c\cos(s)}{r^2+c^2}\ ,\ \frac{r}{r^2+c^2}\right) \\
\kappa(s) &= \frac{r}{r^2+c^2} \\
\tau(s) &= \frac{c}{r^2+c^2}
\end{aligned}$$

The helix therefore has constant curvature and torsion. One can view a circle of radius r as a helix of pitch $c = 0$ and radius r. It then can be seen that a circle has constant curvature $\kappa = 1/r$, and torsion $\tau = 0$ (zero torsion is equivalent to a curve being planar). Similarly, one can view a straight line as helix of infinite radius, and an arbitrary pitch. Taking the limit of $\kappa = \frac{r}{r^2+c^2}$ as r goes to infinity, one can see that a straight line has zero (and therefore constant) curvature.

2.6 Curves in higher dimensions

We generalise the ideas from plane and space curves to a curve in n dimensions, given by $\alpha(t) : \mathbb{R} \mapsto \mathbb{R}^n$, where $\alpha(t) = (x_1(t), x_2(t), \ldots x_n(t))$. Writing $\dot{x}$ for $\frac{dx}{dt}$, the tangent vector is then given by

$$\mathbf{T}(t) = \frac{(\dot{x}_1(t), \dot{x}_2(t), \ldots \dot{x}_n(t))}{\sqrt{\dot{x}_1(t)^2 + \dot{x}_2(t)^2 + \ldots + \dot{x}_n(t)}}$$

To find the curvature at a point, consider the sphere S^n with $(n+1)$-point contact with the curve. Then this sphere has a radius, $1/\kappa$, where κ is the curvature at that point.

The ideas of normal vectors and torsion do not extend so simply. Here, we make a change of notation. Instead of tangent and normal vectors, we use 'frame' vectors $\mathbf{E_1}$, $\mathbf{E_2}$, $\ldots \mathbf{E_n}$. Also, we also use 'generalised curvature' functions $\chi_1 = \kappa$, χ_2, ... up to χ_{n-1}. The frame vectors can be defined inductively, using the Gram-Schmidt orthonormalisation process on the vectors given by derivatives of α. Specifically,

$$\mathbf{E}_1 = \frac{\alpha'(t)}{|\alpha'(t)|} \text{ and } \mathbf{E}_i = \frac{\tilde{\mathbf{E}}_i}{|\tilde{\mathbf{E}}_i|} \text{ for } i > 1$$

$$\text{where } \tilde{\mathbf{E}}_i = \alpha^{(n)}(t) - \sum_{j=1}^{i-1}(\alpha^{(i)}(t).\mathbf{E}_j)\mathbf{E}_j$$

The frame vectors form a right-handed orthonormal basis for $\mathbb{R}^n$. ('Right-handed' means that when the components of the basis are written as successive rows of a square matrix, the determinant of that matrix is +1 rather than -1). In general, the i-dimensional linear subspace that contains the frame vectors $\mathbf{E_1}$ to $\mathbf{E_i}$ has maximal contact with the curve, and further, this subspace is unique in general.

The generalised curvatures χ_i are defined via the n-dimensional Serret-Frenet formulae:

$$\begin{aligned}
\mathbf{E}_1' &= \chi_1 \mathbf{E}_2 \\
\mathbf{E}_i' &= \chi_i \mathbf{E}_{i+1} - \chi_{i-1} \mathbf{E}_{i-1} \\
\mathbf{E}_n' &= -\chi_{n-1} \mathbf{E}_{n-1}
\end{aligned}$$

Alternatively, these formulae can be given in in matrix form:

$$\begin{pmatrix} \mathbf{E_1}' \\ \mathbf{E_2}' \\ \mathbf{E_3}' \\ \vdots \\ \mathbf{E_n}' \end{pmatrix} = \begin{pmatrix} 0 & \chi_1 & 0 & \dots & 0 \\ -\chi_1 & 0 & \chi_2 & \ddots & \vdots \\ 0 & -\chi_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \chi_{n-1} \\ 0 & \dots & 0 & -\chi_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E_1} \\ \mathbf{E_2} \\ \mathbf{E_3} \\ \vdots \\ \mathbf{E_n} \end{pmatrix}$$

In the case $n = 3$, we have that the torsion is χ_2, and the normal and binormal vectors are $\mathbf{E_2}$ and $\mathbf{E_3}$ respectively. It can be shown that if $\chi_{i-1} \neq 0$ and $\chi_i = \chi_{i+1} = \ldots = \chi_{n-1} = 0$, then α is restricted to an i-dimensional linear subspace of $\mathbb{R}^n$. (In the 3D case, if the curvature $\kappa = \chi_1 \neq 0$ and the torsion $\tau = \chi_2 = 0$, then the curve is planar.)

3 Technical tools

This Chapter gives an overview of the techniques and tools that are used in studying the curves from the point of view of singularity theory. Right equivalence provides a means to replace a function with one which is simpler, but which retains the properties that are required. The k-jet of a function does a similar job, providing a polynomial that is a local approximation of a function. The unfolding of a function provides a way to consider a family of functions as a single function instead. Finally, the notion of a curve not being tangent to something is extended to the concept of transversality. This will include manifolds and functions defined on them being transverse to another manifold.

3.1 Right equivalence

Throughout this work, we will be looking at how many derivatives of a function vanish at a particular point. For a curve defined by an arbitrary function, this can lead to manipulation of complicated expressions. Right equivalence is a tool that allows us to use a much simpler function instead, but one that for which the interesting properties are the same.

Suppose we have two curves $f_1(t) : U_1 \subset \mathbb{R} \mapsto \mathbb{R}$ and $f_2(t) : U_2 \subset \mathbb{R} \mapsto \mathbb{R}$, with the subsets U_1, U_2 each containing a point t_1 and t_2 respectively. Then we say that '$f_1(t_1)$ is right equivalent to $f_2(t_2)$' if there exists a diffeomorphism $h : U_1 \mapsto U_2$ such that $f_1 = f_2(h(t))$ with $h(U_1) = U_2$ and $h(t_1) = t_2$. (A diffeomorphism is a smooth bijection with a smooth inverse). We write 'f is right equivalent to g' as $f \sim g$

Right equivalence is an equivalence relation. It is reflexive: taking h as the identity function shows $f \sim f$. It is symmetric: if $f \sim g$ via a diffeomorphism h, then as the inverse of diffeomorphism is also a diffeomorphism, $g \sim f$ via h^{-1}. It is transitive: if $f \sim g$ via a diffeomorphism m, and $g \sim h$ via a diffeomorphism n, then as the composition of two diffeomorphism is a diffeomorphism, we have $f \sim h$ via $m \circ n$.

Right equivalence allows us to say that at a point, a curve is equivalent to another function in all ways that matter. The following theorem from [2] shows that any smooth function will be right-equivalent to a power of t.

Theorem Consider a curve $f(t) : U \subset \mathbb{R} \mapsto \mathbb{R}$, with $f^{(p)}(t_0) = 0$ for $1 \leq p \leq k$ and $f^{(k+1)}(t_0) \neq 0$. Then $f(t_0) \sim \pm t^{k+1}$, with $+$ or $-$ corresponding to $f^{(k+1)}(t_0) > 0$ and < 0 respectively.

3.2 Jets

Consider the Taylor expansion of an (analytic) function about the point $t = t_0$:

$$f(t) = f(t_0) + tf'(t_0) + \frac{t^2}{2!}f''(t_0) + \ldots + \frac{t^s}{s!}f^{(s)}(t_0) + \ldots$$

Then the 'k-jet of f at t_0' is the polynomial

$$j^k f(t_0) = \sum_{r=1}^{k} \frac{t^r f^{(r)} t_0}{r!}$$

This is simply the terms of the Taylor expansion of the function up to and including the kth power, and omitting the constant term. When the constant term is included, it is known as the 'k-jet with constant of f at t_0'. Two k-jets are said to be 'equal' if they are equal as polynomials. A k-jet is useful local polynomial approximation to a given degree of a function. It will used later on to provide a local approximation of a curve that is considered as a graph. In addition, we will look at properties of the co-efficients of the k-jet, and what they translates to geometrically.

3.3 Unfoldings

In section 2.4, we looked at the distance-squared function at a point on the curve. As well as depending on the point $\gamma(t)$, its value also depends on $\mathbf{u}$, the place on the curve the distance is measured from. It is useful to see how the value of the distance-squared function changes as $\mathbf{u}$ varies. Other functions may be defined on a curve that depend on the position on the curve and additional parameters. The function $F : (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$ is called an 'r-parameter unfolding' of f, where $f(t) = F(t, \mathbf{x}_0)$ for some $x_0 \in \mathbb{R}^n$. (For the distance-squared functions, $\mathbf{x} = \mathbf{u}$). Dealing with these functions on an individual basis will, in general, not be simple. This section contains an outline of why any such function will locally look like a polynomial in t, whose co-efficients are functions of $\mathbf{x}$.

Definition Two unfoldings $F(t, \mathbf{x})$ and $G(t, \mathbf{x})$ are equivalent if there exists a diffeomorphism h such that $F = h \circ G$

We now recall the theorem given at the end of section 3.1, which states that any function $f(t)$ which has its first k derivatives vanishing, and the $(k+1)$ derivative non-zero at some $t = t_0$ will be right equivalent to $g(t) = \pm t^{(k+1)}$ at $t = t_0$. Equivalently, $f(t) = g(h(t)) + c$ for t sufficiently close to t_0, a diffeomorphism h, and a constant c.

Now consider the function $\tilde{F}(t, \mathbf{x}) = F(h^{-1}(t), \mathbf{x}) - c$, with $\tilde{F}(t, \mathbf{x_0}) = g(t)$ for some $\mathbf{x} = \mathbf{x_0}$, and for all t sufficiently close to 0. Here, rather than dealing with an arbitrary function f, we are taking a given member of a family of functions parameterised by $\mathbf{x}$, and then saying it is equivalent for some given $\mathbf{x_0}$ to $\pm t^{(k+1)}$

Proposition Consider the function $G(t, \mathbf{x}) = x_1 t + x_2 t^2 + \ldots x_{k-1} t^{k-1} \pm t^{k+1}$. For appropriate diffeomorphisms a, b and c, $\tilde{F}(t, \mathbf{x}) = G(a(t, x), b(x)) + c(x)$.

The main consequence of the proposition is that whenever we have a family of curves with k parameters, and are interested in what happens in the locality of a point, it is is sufficient to use $G(t, \mathbf{x})$ as above instead. When an unfolding is written in the form of $G(t, \mathbf{x})$, then this is known as a 'versal' unfolding. It is also possible to transform an r-parameter unfolding into the form $\tilde{G}(t, \mathbf{x}) = x_1 + x_2 t^1 + x_3 t^2 \ldots x_{r-1} t^{r-2} \pm t^r$. Such a form is known as a 'p-versal unfolding'. A versal unfolding is used when the derivatives of the function are of interest, while a p-versal unfolding is used when the actual values of the function are of interest. We will only be using versal unfoldings, because as we shall see later, it is the derivatives rather than the value of the distance-squared function that can be used to reveal geometric information about a curve.

3.4 Transversality

Definition Consider a smooth function $f(x) : x \in X \subset \mathbb{R}^n \mapsto \mathbb{R}^m$, where X is a smooth manifold and $n < m$; together with a smooth manifold $Y \subset \mathbb{R}^m$ containing $f(x)$. Then f is transverse to Y at the point $f(x)$ if the tangent space to $f(X)$ at $f(x)$ plus the tangent space to Y at $f(x)$ equals the whole of $\mathbb{R}^m$. Further, f is transverse to Y if f is transverse for every $x \in X$, or if $f(X) \cap Y = \oslash$.

Transversality can be thought of as an extension of the idea of two curves being non-tangent. In fact two plane curves are transverse at an intersection point if and only if the two curves are not tangent at that point. However, two curves in n-dimensional Euclidean space ($n > 2$) are transverse only if the curves do not intersect. (If they do intersect, their tangent spaces form a subspace of dimension 2, which cannot span $\mathbb{R}^{n>2}$). A pair of surfaces in $\mathbb{R}^3$ are transverse if they are nowhere tangent to one another. In $\mathbb{R}^n$ ($n \geq 2$), curves are transverse to hypersurfaces of codimension 1 if their intersection set contains only isolated points, and the curve is not tangent to the surface at those points. Points are transverse to a manifold if and only if they are not in that manifold.

4 Singularities

The aim of this section is to define the notion of a type A_k singularity, and then derive in detail what different types of singularity of the distance-squared function correspond to geometrically for plane, space and higher dimensional curves. The bifurcation set of a r-dimensional unfolding is defined, and its local structure is shown to depend only in the type of singularity and r. This information is then used to show the relationship between vertices and cusps on the evolute of a curve.

4.1 Type A_k singularities

Definition Let f be a smooth function. It has a type A_k singularity ($k \geq 0$) at t_0 if the graph of f (i.e. the curve given by $(t, f(t))$) has $k+1$-point contact with the line parallel to the t-axis and passing through the point $(t_0, f(t_0))$.

Equivalently, the graphs of two functions $f(t)$ and $g(t)$ have a $k+1$-point contact at a point $(t_0, f(t_0) = g(t_0))$ when $(f - g)(t)$ has an A_k singularity at t_0. (Setting $g(t) = f(t_0)$ reproduces the above definition).

Any analytic function is of type A_k at a given point for some $k \geq 0$. An analytic function may be defined as one that can be expressed in the form of a convergent power series,

$$f(t) = \sum_{p \geq 0}^{\infty} (a_p (t - t_0))$$

where a_p are real. Taking the smallest p for which $a_{p+1} \neq 0$, we find that f is then of type A_p at the point $t = t_0$. This is equivalent to saying that at an A_k singularity, the first k derivatives vanish, but not the $(k + 1)$th.

It is perfectly possible to have a smooth function which cannot be expressed in the above form, and hence is not analytic. This is the case for any function with $f^{(p)}(t_0) = 0$ for all $p \geq 0$. In this case, the power series expansion of f has co-efficients equal to zero, and thus will not be equal to the original function. For example (from [2]), consider the function:

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & (t > 0) \\ 0 & (t \leq 0) \end{cases}$$

It can be shown that all derivatives of f vanish at $t = 0$, so there is now k for which $f^{(k+1)}(0) \neq 0$, so the function cannot be of type A_k for any k at

$t = t_0$. Functions where all the derivatives vanish at a particular point are known as flat functions. Although a study in their own right, they will not be investigated here. We shall assume that all functions considered are not flat. (For information on flat functions, see [3])

4.2 Bifurcation and singular sets

Let F be an r-parameter family of curves. Then we define the 'singular set', S_F, to be the set of pairs $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$ for which F_x is singular at t:

$$S_F = \{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^r | \frac{\partial F}{\partial t} = 0 \text{ at } (t, x)\}$$

We also define the 'bifurcation set' $\mathcal{B}_F$ of F to be:

$$\mathcal{B}_F = \{x \in \mathbb{R}^r | \text{ there exists } t \text{ such that } \frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0 \text{ at } (t, x)\}$$

The main interest of the bifurcation set lies in its local structure. We will see later that whether the bifurcation set of the distance-squared function is locally a cusp or a line tells us where a curve has a vertex.

If the unfolding is an r-parameter versal unfolding, and has an A_k singularity at a point, then the local structure bifurcation set at that point is determined solely by the values of r and k, up to a diffeomorphism. However, for an A_k singularity to occur, the inequality $r \geq k - 1$ must hold. Further, we will see that if we denote the bifurcation set for the case $r = k - 1$ by B_r, then the bifurcation set in general will be $B_r \times \mathbb{R}^{r-k+1}$. It is illustrative to compute the bifurcation set for the first few values of k.

A_2 singularity If we have an A_2 singularity, then we must have at least 1-parameter unfolding. For a 1-parameter versal unfolding we have $F(t, x) = t^3 + xt$. Then the bifurcation set

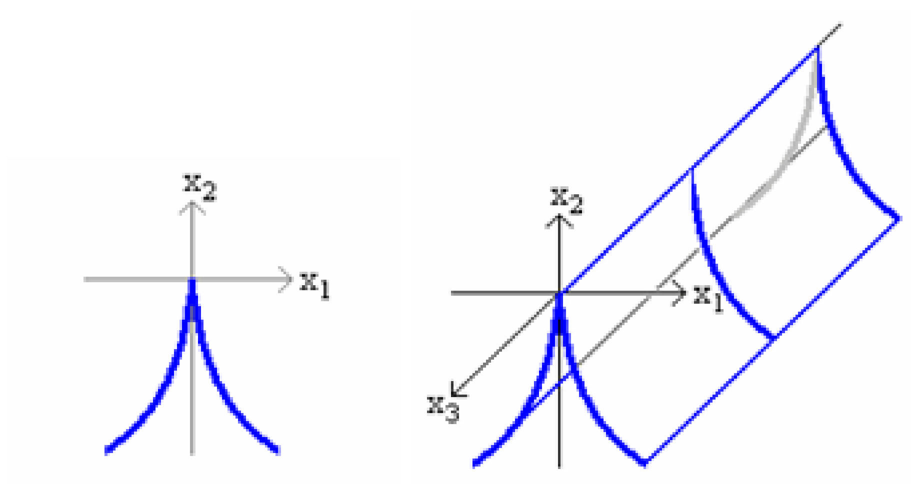
$$\mathcal{B}_F = \{x \in \mathbb{R} | \text{ there exists } t \text{ such that } 3t^2 + x = 6t = 0 \text{ at } (t, x)\} = \{0\}$$

Hence $\mathcal{B}_F$ is a single point. When we have more than one parameter, then we instead have $\tilde{F}(t, \mathbf{x}) = t^3 + x_1 t$, independent of $x_2 \ldots x_r$. Then $\mathcal{B}_{\tilde{F}} = \mathcal{B}_F \times \mathbb{R}^{r-1}$. This is a line for $r = 2$, a plane for $r = 3$, and an $(r-1)$-dimensional linear subspace in general (which is just $\{0\} \times \mathbb{R}^{r-1}$).

A_3 singularity If we have an A_3 singularity, then we must have at least 2-parameter unfolding. For a 2-parameter versal unfolding we have $F(t,x) = t^4 + x_1t^2 + x_2t$. Then the bifurcation set is

$$\begin{aligned}
\mathcal{B}_F &= \{x \in \mathbb{R} |\ \exists t \text{ such that } 4t^3 + 2x_1t + x_2 = 12t^2 + 2x_1 = 0 \text{ at } (t,x)\} \\
&\Rightarrow x_1 = -6t^2 \\
&\Rightarrow x_2 = 8t^3 \\
\Rightarrow \mathcal{B}_F &= \{\mathbf{x} \,|\, x_1 = -6t^2 \ ,\ x_2 = 8t^3 \text{ for some } t\} \\
\Rightarrow \mathcal{B}_F &= \{\mathbf{x} \,|\, 8x_1 + 27x_2 = 0\}
\end{aligned}$$

This is a cusp, which is shown in Figure 4(a), and we shall denote it by B_2. For $r >$, then by similar arguments used for the A_2 case, we find that the bifurcation set is $B_2 \times \mathbb{R}^{r-2}$. Figure 4(b) shows $B_2 \times \mathbb{R}$, known as the cuspoidal edge.



(a) Cusp (b) Cuspoidal edge

Figure 4: Cusp and cuspoidal edge

Higher singularities We have seen in two cases above that when an r-parameter unfolding has an A_k singularity, the result is a $(r-1)$-dimensional subspace of $\mathbb{R}^r$. This trend continues, and we find that the bifurcation set of 3-parameter versal unfolding at an A_4 singularity is a surface known as the swallowtail surface, shown in Figure 5. The swallowtail surface can be given by the zero set of a fourth degree polynomial. Further, when a 4-parameter unfolding has an A_5 singularity, the bifurcation set a is three dimensional hypersurface known as the butterfly surface.

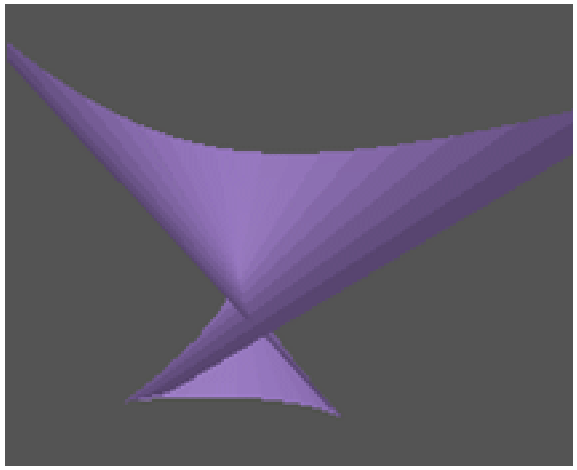


Figure 5: Swallowtail surface

To summarise, the bifurcation set of a r-parameter unfolding with an A_k singularity ($k \leq r+1$) is $B_r \times \mathbb{R}^{r-k+1}$, where the form of each B_i is as follows:

- $B_1 = \{0\}$, a point, and occurs with an A_2 singularity.
- $B_2 = \{(x_1, x_2) | x_1^2 = x_2^3\}$, a cusp (Figure 4(a) on page 22)
- $B_3 = \{(x_1, x_2, x_3) | x_1^2 = 3u^4 + u^2 x_3, x_2 = 4u^3 + 2ux_3\}\}$, the swallowtail surface (Figure 5 on page 23), and occurs with s an A_4 singularity.
- B_4 is the three dimensional butterfly hypersurface, and occurs when a 4-parameter unfolding has an A_5 singularity.

4.3 Singularities of the distance-squared function

In this section we look at the geometric consequences of the distance-squared function having an A_k singularity, starting with plane curves, and then extending to space curves and higher dimensions.

4.3.1 Plane curves

Considering the derivatives of the distance squared function, using the Serret-Frenet formulae and remembering that $\mathbf{T}.\mathbf{T}' = \mathbf{N}.\mathbf{N}' = \mathbf{T}.\mathbf{N} = 0$, and$\gamma' = \mathbf{T}$, we find the following:

$$\begin{aligned}
\mathbf{D_u(t)} &= (\gamma - \mathbf{u}).(\gamma - \mathbf{u}) \\
\mathbf{D'_u(t)} &= 2\gamma'.(\gamma - \mathbf{u}) \\
&= 2\mathbf{T}.(\gamma - \mathbf{u}) \\
\mathbf{D''_u(t)} &= 2\mathbf{T}'.(\gamma - \mathbf{u}) + 2\mathbf{T}.\gamma' \\
&= 2\kappa\mathbf{N}.(\gamma - \mathbf{u}) + 2 \\
\mathbf{D^{(3)}_u(t)} &= 2\kappa'\mathbf{N}.(\gamma - \mathbf{u}) + 2\kappa\mathbf{N}'.(\gamma - \mathbf{u}) + 2\kappa\mathbf{N}.\gamma' \\
&= 2\kappa'\mathbf{N}.(\gamma - \mathbf{u}) - 2\kappa^2\mathbf{T}.(\gamma - \mathbf{u}) + 2\kappa\mathbf{N}.\mathbf{T} \\
&= 2\kappa'\mathbf{N}.(\gamma - \mathbf{u}) - 2\kappa^2\mathbf{T}.(\gamma - \mathbf{u})
\end{aligned}$$

We then consider the geometric consequences of these derivatives vanishing at a point γt_0

$$\begin{aligned}
\mathbf{D'_u(t)} = 0 &\Leftrightarrow 0 = 2\mathbf{T}.(\gamma - \mathbf{u}) \\
&\Leftrightarrow \mathbf{T} \text{ is perpendicular to } (\gamma - \mathbf{u}) \\
&\Leftrightarrow (\gamma - \mathbf{u}) = \lambda\mathbf{N}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D'_u(t)} = \mathbf{D''_u(t)} = 0 &\Leftrightarrow 0 = 2\kappa\mathbf{N}.(\gamma - \mathbf{u}) + 2 \\
&\Leftrightarrow 0 = 2\kappa\mathbf{N}.\lambda\mathbf{N} + 2 \\
&\Leftrightarrow 0 = 2\kappa\lambda + 2 \Leftrightarrow \kappa \neq 0 \\
&\Leftrightarrow \lambda = -\frac{1}{\kappa} \\
&\Leftrightarrow \mathbf{u} = \gamma + \frac{\mathbf{N}}{\kappa}
\end{aligned}$$

$$\begin{aligned}
\mathbf{D'_u(t)} = \mathbf{D''_u(t)} = \mathbf{D^{(3)}_u(t)} = 0 &\Leftrightarrow 0 = 2\kappa'\mathbf{N}.(\gamma - \mathbf{u}) - 2\kappa^2\mathbf{T}.(\gamma - \mathbf{u}) \\
&\Leftrightarrow 0 = 2\kappa'\mathbf{N}.(-\frac{\mathbf{N}}{\kappa}) - 2\kappa^2\mathbf{T}.(-\frac{\mathbf{N}}{\kappa}) \\
&\Leftrightarrow 0 = -2\frac{\kappa'}{\kappa}
\end{aligned}$$

$$\Leftrightarrow \quad \kappa' = 0, \kappa \neq 0, \mathbf{u} = \gamma + \frac{\mathbf{N}}{\kappa}$$

To summarise, if the distance-squared function has an A_1 singularity, $\mathbf{u}$ is on the normal line to γ at $\gamma(t_0)$; an A_2 singularity additionally means the curvature is non-zero and that $\mathbf{u}$ is at the centre of curvature; an A_3 singularity additionally means γ has an ordinary vertex at that point and similar arguments show that an A_4 singularity additionally means γ has a higher vertex at that point.

4.3.2 Space curves

We shall now look at extending the distance-squared function to space curves. It is defined in the same way as for planar curves:

$$D_{\mathbf{u}}(t) = |\ \alpha(t) - \mathbf{u}\ |^2 = (\alpha(t) - \mathbf{u}).(\alpha(t) - \mathbf{u})$$

As for planar curves, we will look at the function's derivatives, and then consider the geometric consequences.

$$\begin{aligned}
\mathbf{D}'_{\mathbf{u}}(t) &= 2\alpha'.(\alpha - \mathbf{u}) \\
&= 2\mathbf{T}.(\alpha - \mathbf{u}) \\
\mathbf{D}''_{\mathbf{u}}(t) &= 2\mathbf{T}'.(\alpha - \mathbf{u}) + 2\mathbf{T}.\alpha' \\
&= 2\kappa\mathbf{N}.(\alpha - \mathbf{u}) + 2 \\
\mathbf{D}^{(3)}_{\mathbf{u}}(t) &= 2\kappa'\mathbf{N}.(\alpha - \mathbf{u}) + 2\kappa\mathbf{N}'.(\alpha - \mathbf{u}) + 2\kappa\mathbf{N}.\alpha' \\
&= 2\kappa'\mathbf{N}.(\alpha - \mathbf{u}) + 2\kappa(\tau\mathbf{B} - \kappa\mathbf{T}).(\alpha - \mathbf{u}) + 2\kappa^2\mathbf{T}.\mathbf{T} \\
&= 2\kappa'\mathbf{N}.(\alpha - \mathbf{u}) + 2\kappa(\alpha - \mathbf{u}).(\tau\mathbf{B} - \kappa\mathbf{T})
\end{aligned}$$

We now consider the consequences of these derivatives vanishing at a point, remembering that the tangent, normal and binormal vectors are of length one, and are perpendicular to each other and to their own derivatives. γt_0

$$\begin{aligned}
\mathbf{D}'_{\mathbf{u}}(t) = 0 &\Leftrightarrow 0 = 2\mathbf{T}.(\alpha - \mathbf{u}) \\
&\Leftrightarrow \mathbf{T} \text{ is perpendicular to } (\alpha - \mathbf{u})
\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \quad (\alpha - \mathbf{u}) = \lambda \mathbf{N} + \mu \mathbf{B} \text{ for some } \lambda, \mu \in \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}_{\mathbf{u}}''(t) = \mathbf{D}_{\mathbf{u}}'(t) = 0 \Rightarrow 0 &= 2\kappa \mathbf{N}.(\alpha - \mathbf{u}) + 2 \\
&= 2\kappa \mathbf{N}.(\lambda \mathbf{N} + \mu \mathbf{B}) + 2 \\
&= 2\kappa\lambda \mathbf{N}.\mathbf{N} + 2\kappa\mu \mathbf{N}.\mathbf{B} + 2 \\
&= 2\kappa\lambda + 2 \Rightarrow \kappa \neq 0 \\
&\Leftrightarrow \lambda = -\frac{1}{\kappa} \\
&\Leftrightarrow (\alpha - \mathbf{u}) = \mu \mathbf{B} - \frac{\mathbf{N}}{\kappa} \text{ for some } \mu \in \mathbb{R}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}_{\mathbf{u}}^{(3)}(t) = \ldots = \mathbf{D}_{\mathbf{u}}'(t) = 0 & \\
\Rightarrow 0 &= 2\kappa' \mathbf{N}.(\alpha - \mathbf{u}) + 2\kappa(\alpha - \mathbf{u}).(\tau \mathbf{B} - \kappa \mathbf{T}) \\
&= 2\kappa' \mathbf{N}.(\mu \mathbf{B} - \frac{\mathbf{N}}{\kappa}) + 2\kappa(\mu \mathbf{B} - \frac{\mathbf{N}}{\kappa}).(\tau \mathbf{B} - \kappa \mathbf{T}) \\
&= 2\kappa' \frac{-1}{\kappa} + 2\kappa\mu\tau \\
\Leftrightarrow 0 &= -\kappa' + \kappa^2 \mu\tau \ (\kappa \neq 0) \\
\text{if } \tau = 0 &\Rightarrow \kappa' = 0 \ , \ \mu \in \mathbb{R} \\
\text{if } \tau \neq 0 &\Rightarrow \mu = \frac{\kappa'}{\kappa^2 \tau}
\end{aligned}
$$

To summarise, if the distance-squared function has an A_1 singularity, $\mathbf{u}$ is in the normal plane to γ at $\gamma(t_0)$; an A_2 singularity additionally means the curvature is non-zero, and $\mathbf{u}$ is on the line through the centre of curvature and parallel to the binormal; and an A_3 singularity additionally means that either the torsion is zero and γ has a vertex at $\gamma(t_0)$, or that the torsion is non-zero and $\mathbf{u} = \alpha - \frac{\kappa'}{\kappa^2\tau}\mathbf{B} + \frac{\mathbf{N}}{\kappa}$.

4.3.3 Curves in higher dimensions

Once again, we extend the ideas and techniques from two and three dimensions into an arbitrary higher number of dimensions. We start by stating the derivatives of the distance squared function:

$$
\mathbf{D}_{\mathbf{u}}'(t) = 2\mathbf{E}_1.(\alpha - \mathbf{u})
$$

$$
\begin{aligned}
\mathbf{D}''_{\mathbf{u}}(t) &= 2\chi_1\,\mathbf{E}_2.(\alpha-\mathbf{u})+2\\
\mathbf{D}^{(3)}_{\mathbf{u}}(t) &= -2\chi_1^2\,\mathbf{E}_1.(\alpha-\mathbf{u})+2\chi_1'\,\mathbf{E}_2.(\alpha-\mathbf{u})+2\chi_1\chi_2\,\mathbf{E}_3.(\alpha-\mathbf{u})\\
\mathbf{D}^{(4)}_{\mathbf{u}}(t) &= (-2\chi_1^2-6\chi_1\chi_1')\,\mathbf{E}_1.(\alpha-\mathbf{u})\\
&\quad+(2\chi_1''-2\chi_1\chi_1'-2\chi_1\chi_2^2)\,\mathbf{E}_2.(\alpha-\mathbf{u})\\
&\quad+(2\chi_1\chi_2')\,\mathbf{E}_3.(\alpha-\mathbf{u})+2\chi_1\chi_2\chi_3\,\mathbf{E}_4.(\alpha-\mathbf{u})-2\chi_1^2
\end{aligned}
$$

Generalising, we have for $r \geq 3$ and $n \geq 3$:

$$
\mathbf{D}^{(r)}_{\mathbf{u}}(t) = \left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).(\alpha-\mathbf{u})+\psi_0^r
$$
$$
\text{Where } \psi_i^r = 0 \text{ for } i > r
$$

$\mathbf{D}^{(r)}_{\mathbf{u}}(t)$ can always be written in the form shown above when ψ is a function of t. (In fact, ψ is a polynomial in χ_i and their derivatives). The only possible problem is the requirement that $\psi_i^r = 0$ for $i > r$, and we shall prove this inductively. Clearly it is true for the cases $r = 3$ and $r = 4$. So, all that needs to be done is to show that if $\psi_i^r = 0$ for $i > r$, then $\psi_i^{r+1} = 0$ for $i > r+1$. We shall find ψ_i^{r+1} in terms of ψ_i^r. Writing x' for $\frac{dx}{dt}$, we have:

$$
\begin{aligned}
\mathbf{D}^{(r)}_{\mathbf{u}}(t) &= \left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).(\alpha-\mathbf{u})+\psi_0^r\\
\mathbf{D}^{(r+1)}_{\mathbf{u}}(t) &= \left(\sum_{i=1}^{n}(\psi_i^r)'\mathbf{E}_i\right).(\alpha-\mathbf{u})+\left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i'\right).(\alpha-\mathbf{u})\\
&\quad+\left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).\alpha'+(\psi_0^r)'\\
&= \left(\sum_{i=1}^{n}(\psi_i^r)'\mathbf{E}_i\right).(\alpha-\mathbf{u})+\left(\sum_{i=2}^{n}-\psi_i^r\chi_{i-1}\mathbf{E}_{i-1}\right).(\alpha-\mathbf{u})\\
&\quad+\left(\sum_{i=1}^{n-1}\psi_i^r\chi_i\mathbf{E}_{i+1}\right).(\alpha-\mathbf{u})+\left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).\mathbf{E}_1+(\psi_0^r)'\\
&= \left(\sum_{i=1}^{n}(\psi_i^r)'\mathbf{E}_i\right).(\alpha-\mathbf{u})+\left(\sum_{i=1}^{n-1}-\psi_{i+1}^r\chi_i\mathbf{E}_i\right).(\alpha-\mathbf{u})\\
&\quad+\left(\sum_{i=2}^{n}\psi_{i-1}^r\chi_{i-1}\mathbf{E}_i\right).(\alpha-\mathbf{u})+\psi_1^r+(\psi_0^r)'
\end{aligned}
$$

So, equating co-efficients, we have (for $r > 3$):

$$\begin{aligned}
\psi_0^{r+1} &= (\psi_0^r)' + \psi_1^r \\
\psi_1^{r+1} &= (\psi_1^r)' - \chi_1\psi_2^r \\
\psi_i^{r+1} &= (\psi_i^r)' - \chi_i\psi_{i+1}^r + \chi_{i-1}\psi_{i-1}^r \quad \text{for } 1 < i < n \\
\psi_n^{r+1} &= (\psi_n^r)' + \chi_{n-1}\psi_{n-1}^r
\end{aligned}$$

This also implies the useful:

$$\psi_j^j = 2\prod_{i=1}^{j-1}\chi_i \quad \text{for } j \leq n$$

Then we see that $\psi_i^{r+1} = 0$ when $i > r+1$, providing $r+1 < n$ as required.

The next stage is establish what happens when successive derivatives vanish. We shall use the fact that we can write that $(\alpha - \mathbf{u}) = \sum_{i=1}^n \lambda_i \mathbf{E}_i$ with $\lambda_i \in \mathbb{R}$.

$$\begin{aligned}
\mathbf{D}'_{\mathbf{u}}(t) = 0 \quad &\Leftrightarrow \quad 0 = 2\alpha'.(\alpha - \mathbf{u}) \\
&\Leftrightarrow \quad 0 = 2\mathbf{E}_1.(\alpha - \mathbf{u}) \\
&\Leftrightarrow \quad \mathbf{E}_1 \text{ is perpendicular to } (\alpha - \mathbf{u}) \\
&\Leftrightarrow \quad (\alpha - \mathbf{u}) = \sum_{i=2}^n \lambda_i \mathbf{E}_i \ , \ \lambda_i \in \mathbb{R} \\
&\Rightarrow \quad \lambda_1 = 0 \\
\mathbf{D}''_{\mathbf{u}}(t) = \mathbf{D}'_{\mathbf{u}}(t) = 0 \Rightarrow 0 \quad &= \quad 2\chi_1\mathbf{E}_2.(\alpha - \mathbf{u}) + 2 \\
&= \quad 2\chi_1\mathbf{E}_2.(\sum_{i=2}^n \lambda_i \mathbf{E}_i) + 2 \\
&= \quad 2\chi_1\lambda_2\mathbf{E}_2.\mathbf{E}_2 + \chi_1\mathbf{E}_2.(\sum_{i=3}^n \lambda_i \mathbf{E}_i) + 2 \\
&= \quad 2\chi_1\lambda_2 + 2 \Rightarrow \chi_1 \neq 0 \\
&\Leftrightarrow \quad \lambda_2 = -\frac{1}{\chi_1} \\
&\Leftrightarrow \quad (\alpha - \mathbf{u}) = \frac{\mathbf{E}_2}{\chi_1} + \sum_{i=3}^n \lambda_i \mathbf{E}_i
\end{aligned}$$

$$
\begin{aligned}
\mathbf{D}^{(3)}_{\mathbf{u}}(t) = \mathbf{D}''_{\mathbf{u}}(t) = \mathbf{D}'_{\mathbf{u}}(t) = 0 \Rightarrow 0 &= -2\chi_1^2\,\mathbf{E}_1.(\alpha - \mathbf{u}) + 2\chi_1'\,\mathbf{E}_2.(\alpha - \mathbf{u}) \\
&\quad +2\chi_1\chi_2\,\mathbf{E}_3(\alpha - \mathbf{u}) \\
&= -2\chi_1^2\,\mathbf{E}_1.\left(\frac{\mathbf{E}_2}{\chi_1} + \sum_{i=3}^{n}\lambda_i\mathbf{E}_i\right) \\
&\quad +2\chi_1'\,\mathbf{E}_2.\left(\frac{\mathbf{E}_2}{\chi_1} + \sum_{i=3}^{n}\lambda_i\mathbf{E}_i\right) \\
&\quad +2\chi_1\chi_2\,\mathbf{E}_3.\left(\frac{\mathbf{E}_2}{\chi_1} + \sum_{i=3}^{n}\lambda_i\mathbf{E}_i\right) \\
&= -2\frac{\chi_1'}{\chi_1} + 2\chi_1\chi_2\lambda_3 \\
\Leftrightarrow 0 &= \chi_1' - \chi_1^2\chi_2\lambda_3 \\
\text{if } \chi_2 = 0 &\Rightarrow \chi_1' = 0\ ,\ \lambda_3 \in \mathbb{R}\ \text{(plane curve, vertex)} \\
\text{if } \chi_2 \neq 0 &\Rightarrow \lambda_3 = \frac{\chi_1'}{\chi_1^2\chi_2} \\
&\Rightarrow (\alpha - \mathbf{u}) = \frac{\mathbf{E}_2}{\chi_1} + \frac{\chi_1'\mathbf{E}_3}{\chi_1^2\chi_2} + \sum_{i=4}^{n}\lambda_i\mathbf{E}_i
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D}^{(r)}_{\mathbf{u}}(t) = \ldots = \mathbf{D}'_{\mathbf{u}}(t) = 0 & \\
\text{Assume } \chi_i \neq 0 \text{ for } i \leq r-2 & \\
\Rightarrow 0 &= \left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).(\alpha - \mathbf{u}) + \psi_0^r \\
&= \left(\sum_{i=1}^{n}\psi_i^r\mathbf{E}_i\right).\left(\sum_{j=1}^{n}\lambda_j\mathbf{E}_j\right) + \psi_0^r \\
&= \sum_{i=1}^{n}(\psi_i^r\lambda_i) + \psi_0^r \\
\text{If } r = n \Rightarrow 0 &= \sum_{i=1}^{r-1}(\psi_i^r\lambda_i) + \psi_r^r\lambda_r + \psi_0^r \\
\text{If } r < n \Rightarrow 0 &= \sum_{i=1}^{r-1}(\psi_i^r\lambda_i) + \psi_r^r\lambda_r + \sum_{i=r+1}^{n}(\psi_i^r\lambda_i) + \psi_0^r \\
&= \sum_{i=1}^{r-1}(\psi_i^r\lambda_i) + \psi_r^r\lambda_r + \psi_0^r
\end{aligned}
$$

$$
\begin{aligned}
\text{Now } r \leq n \quad &\Rightarrow \quad \psi_r^r = 2\prod_{i=1}^{r-1}\chi_i \\
r \leq n \text{ and } \chi_{r-1} \neq 0 \Rightarrow \psi_r^r \neq 0 \quad &\Rightarrow \quad \lambda_r = -\frac{\psi_0^r}{\psi_r^r} - \sum_{i=1}^{r-1}\left(\frac{\psi_i^r\lambda_i}{\psi_r^r}\right) \\
r \leq n \text{ and } \chi_{r-1} = 0 \Rightarrow \psi_r^r = 0 \quad &\Rightarrow \quad \sum_{i=1}^{r-1}(\psi_i^r\lambda_i) + \psi_0^r = 0 \text{ and } \lambda_r \in \mathbb{R} \\
\text{If } r > n \Rightarrow 0 \quad &= \quad \sum_{i=1}^{n}(\psi_i^r\lambda_i) + \psi_0^r
\end{aligned}
$$

Recall that $(\alpha - \mathbf{u}) = \sum_{i=1}^{n}\lambda_i\mathbf{E}_i$. When $\mathbf{D_u}(t)$ has at least an A_k singularity with $k \leq n$, there are two possibilities. Either $\mathbf{u}$ is confined to a linear subspace of codimension k; or the curve α is confined to a linear subspace of dimension $k-1$. In the former case, the subspace is given by $\{\mathbf{v} \in \mathbb{R}^n | \mathbf{v} = \sum_{i=1}^{k}(\lambda_i\mathbf{E}_i) + \text{span}\{\mathbf{E}_{k+1}\ ,\ \ldots\ ,\ \mathbf{E}_n\}\}$, with λ_i fixed using the formula above. In the latter case, we have the additional restriction that $\sum_{i=1}^{r-1}(\psi_i^r\lambda_i) + \psi_0^r = 0$. As each ψ can be given as polynomial in the χ_is and their derivatives, this equates to a rather complicated restriction on the χ_is and their derivatives.

When $\mathbf{D_u}(t)$ has at least an A_k singularity with $k > n$, then $\mathbf{u}$ must be a point given by $\mathbf{u} = \alpha - \sum_{i=1}^{n}\lambda_i\mathbf{E}_i$. Further, $\psi_0^r + \sum_{i=1}^{n}\psi_i^r\lambda_i = 0$, which again equates to a rather complicated restriction on the χ_i and their derivatives.

4.4 Bifurcation set of the distance-squared function

Now that the geometric consequences of the distance-squared function having an A_k singularity at a point have been established, the next stage is to look at the bifurcation set, and in particular, its local structure.

4.4.1 Plane curves

For plane curves, the distance-squared function can be given as a 2-parameter unfolding. We saw in section 4.2 that the bifurcation set of such a function is locally diffeomorphic to a cusp when it has at least an A_3 singularity, and a line otherwise. Recall that the evolute is given by $\xi(t) = \gamma(t) + \frac{\mathbf{N}(t)}{\kappa(t)}$, and that the bifurcation set of a function $F(t, \mathbf{x})$ is given by $\mathcal{B}_F = \{x \in \mathbb{R}^r |$ there exists t such that $\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial t^2} = 0$ at $(t, x)\}$. Taking $F(t, \mathbf{x}) = D_\mathbf{u}(t)$

(where $\mathbf{x} = \mathbf{u}$), we obtain that:

$$\mathcal{B}_D = \{\mathbf{u} \in \mathbb{R}^2 | \text{ there exists } t \text{ such that } D'_{\mathbf{u}}(t) = D''_{\mathbf{u}}(t) = 0 \text{ at } (t, \mathbf{u})\}$$

We know from section 4.3.1 that $D'_{\mathbf{u}}(t) = D''_{\mathbf{u}}(t) = 0 \Rightarrow \mathbf{u} = \gamma(t) + \frac{\mathbf{N}(t)}{\kappa(t)}$ i.e., $\mathbf{u}$ is on the evolute of γ. Thus the bifurcation set of the distance-squared function of a plane curve is exactly the evolute of the curve. We also know that when the $D_{\mathbf{u}}(t)$ has a least an A_3 singularity, then $\gamma(t)$ has a vertex. Therefore the evolute is locally diffeomorphic to a line, except at the points corresponding to vertices, where the evolute has a cusp. There is an implicit assumption here that the vertices of $\gamma(t)$ are isolated, and we will see in the next Chapter that this is the case for 'almost all' curves.

4.4.2 Curves in three and higher dimensions

As before, the bifurcation set of the distance-squared function on an n-dimensional curve is given by:

$$\mathcal{B}_D = \{\mathbf{u} \in \mathbb{R}^2 | \text{ there exists } t \text{ such that } D'_{\mathbf{u}}(t) = D''_{\mathbf{u}}(t) = 0 \text{ at } (t, \mathbf{u})\}$$

From sections 4.3.2 and 4.3.3, we know that $D'_{\mathbf{u}}(t) = D''_{\mathbf{u}}(t) = 0 \Rightarrow \mathbf{u} = \alpha(t) - \frac{\mathbf{E}_2}{\chi_1} + \sum_{i=3}^{n} \lambda_i \mathbf{E}_i$ i.e. $\mathbf{u}$ is on a surface of codimension 1 which contains the evolute of γ. We shall refer to this surface as the 'bifurcation sheet'. We also know that when the $D_{\mathbf{u}}(t)$ has an A_k singularity with $k \geq 3$, then either $\gamma(t)$ has a vertex, or $\chi_2 = 0$. Further, the bifurcation sheet will be locally diffeomorphic to $B_{k-1} \times \mathbb{R}^{n-k+1}$, and as $k \geq 3$, this will be diffeomorphic to something other than $\mathbb{R}^{n-1}$. Thus the evolute being locally diffeomorphic to something other than a straight line indicates the presence of a vertex or a point where $\chi_2 = 0$.

5 Generic properties

In this section we seek to establish what local properties of a curve remain when the curve is perturbed slightly. Such properties are called 'generic', and this section opens with a rigorous definition of this concept. The main tool that is used to establish what properties are generic is the Monge-Taylor map, which maps a plane curve to the real vector space consisting of the coefficients of a particular polynomial approximation to a given degree, which yields conditions for the existence of vertices and inflexions, both higher and ordinary.

We then use the transversality theorem to show that, in general, a plane curve has only finitely many ordinary vertices and inflections, and no higher vertices or inflections.

5.1 Monge-Taylor map

In this section, we will nominally restrict ourselves to closed plane curves (those of the form $\gamma : t \in S^1 \mapsto \mathbb{R}^2$). This may seem a major restriction, but we shall see later that the results that follow can be easily applied to open curves as well.

Consider a smooth closed curve containing a point $\gamma(t_0)$. Then there will exist some neighborhood $U \subset \mathbb{R}$ of t_0, such that $\gamma(t)$ is a graph of t for all $t \in U$ with co-ordinate axes being the tangent and normal lines at $\gamma(t_0)$, for some function (see Figure 6 on 33). We denote this function by $\eta = f_t(\zeta)$, where η is in the direction of the normal vector at $\gamma(t_0)$, and ζ is the direction of the tangent vector. This form is known as the 'Monge normal form'. If t is replaced through some reparameterisation $s = h(t)$, then $f_t = f_s$. This means that the Monge normal form depends solely on the geometry of the curve, rather than any particular parameterisation.

Definition The 'Monge-Taylor map of order k' $\mu_\gamma(t) : t \in I \mapsto V_k$ is the map that associates at each $t \in I$ the k-jet of the function f_t at 0 - that is, the Taylor series of f_t about zero truncated up to the t^k term, and omitting the constant term. More simply, $\mu_\gamma(t) = j^k f_t(0)$.

In general, μ will be used rather than μ_γ. We will identify V_k with $\mathbb{R}^{k-1}$, so that the co-efficients of the Taylor series (which will depend on t) are written in the form $(a_2(t), a_3(t), \ldots, a_k(t))$, where $a_l(t) = f_t^{(l)}(0)/l!$. Note that as

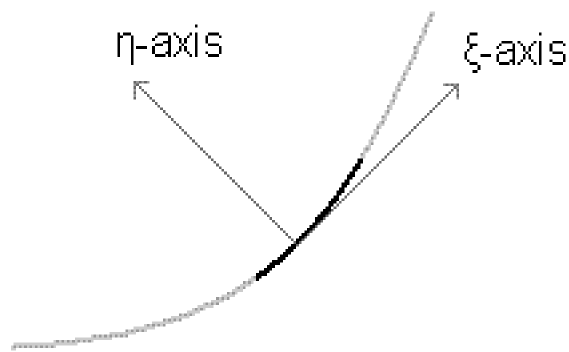


Figure 6: Monge normal form axes

the ζ axis is tangent at $\gamma(t)$ then $a_0(t) = f_t(0) = 0$ and $a_1(t) = f_t'(0) = 0$, so the image of γ will always be in V_k.

Conditions for vertices and inflexions Denoting $df/d\zeta$ by f', it can be shown that $\kappa(t) = 2a_2(t)$, $\kappa'(t) = 6a_3(t)$ and $\kappa''(t) = 24(a_4(t) - a_2(t)^3)$. Then $\gamma(t)$ has

- an inflexion iff $a_2 = 0$
- a higher inflexion iff $a_2 = a_3 = 0$
- a vertex iff $a_2 \neq 0$ and $a_3 = 0$;
- a higher vertex iff $a_2 \neq 0$ and $a_3 = a_4 - a_2^3 = 0$

It is a classical result that a plane curve can be uniquely defined by its curvature, given a starting point and direction. We have just seen how points in the Monge-Taylor image μ_γ correspond to details about the curvature. Given this, it is worth considering the extent to which a curve is determined by its Monge-Taylor image. If we take the curve as being unit speed, then it can be shown that its image in V_3 is sufficient; otherwise its image in V_4 is needed to determine the curve up to a starting point and direction.

The image of a plane curve γ under the Monge-Taylor map of order 4 in $\mathbb{R}^3$ will be a space curve. The above conditions for $\gamma(t)$ to have a higher or ordinary inflexion or vertex mean that μ_γ must intersect the curves or surfaces (which are all smooth manifolds) described by those conditions. As stated in section 3.4, the space curve μ_γ will be transverse to the curves corresponding to higher vertices and inflexions if and only if they don't intersect. Further, μ_γ will be transverse to the surface for (general) vertices and inflexions if and only if it intersects at isolated points and is not tangent to the surface at those points. Thus μ_γ being transverse to these surface and points is equivalent to saying γ has no higher vertices or higher inflexions and only isolated ordinary vertices and ordinary inflexions. Geometric intuition suggests that μ_γ is 'almost always' transverse to the curves and surfaces. In the next section, the notion of 'almost always' is given a more formal definition.

5.2 Open and dense properties

Definition Consider the Euclidean space $\mathbb{R}^n$, and an associated property such that any point in $\mathbb{R}^n$ either possesses that property or does not. The property is said to be 'open' if for any point $p \in \mathbb{R}^n$ there exists a neighborhood $U \subset \mathbb{R}^n$ of p such that U contains a point with that property. The property is said to be 'dense' if given a point p with the property, there exists a neighborhood $U \subset \mathbb{R}^n$ of p such that all points in U have that property. A property that is both open and dense is called 'generic'.

An example of a generic property with $\mathbb{R}^n$ would be 'p is not a member of a subset of zero measure'. (Any subspace of $\mathbb{R}^n$ of dimension strictly less than n will be a set of zero measure; and any intersection or finite union of sets of zero measure is also a set of zero measure). Informally, this is because one can always take a point on a set of zero measure and move it an arbitrarily small distance so that it is not a member of the set (so the property is open). Further, given a point not on a set of zero measure, we can the find the smallest distance to any set of zero measure, take an open ball round the point with radius half that distance as a neighborhood, and then any point in that neighborhood will not be a member of the set of zero measure (so the property is dense). Thus the property is generic. This example will be used later on.

Example: inflexions on the cubic curve Another example relates more closely to curves. Consider a cubic polynomial $y = P(x)$. Then the property 'P(x) has no points of inflexion' is generic. To see this, first note that any cubic curve can be written in the form $y = x^3 + ax + b$, following a suitable linear transformation of co-ordinates. Thus any cubic curve can be thought of as a point $(a, b) \in \mathbb{R}^2$. For (x, y) to be a point of inflexion, then $P'(x) = P''(x) = 0$. Now $P'(x) = 0 \Rightarrow 3x^2 + x = 0$, and $P''(x) = 0 \Rightarrow 6x = 0 \Rightarrow$

$x = 0 \Rightarrow a = 0$. Thus the set of all cubics with an inflexion is the set $\{ (a, b) \in \mathbb{R}^2 : a = 0 \}$. As this is a line and has dimension one, it forms a set of zero measure in $\mathbb{R}^2$, which is our space of all cubic curves. Thus by the example in the previous paragraph, '$P(x)$ has no points of inflexion' is a generic property.

It is possible to extend the notion of open and dense properties to spaces of functions, so that we are able to talk about generic properties of a function. This is a fairly small jump, as any analytic function can be expressed as a polynomial via its Taylor series, and the vector space of polynomials has a natural identification with $\mathbb{R}^n$. As the space of all smooth functions is of infinite dimension, this explanation is purely an informal one.

5.3 Transversality theorem

The transversality theorem is as follows: Let X be compact and $Y \subset \mathbb{R}^m$ be a smooth manifold which is a closed subset of $\mathbb{R}^m$. Consider the set of smooth mappings $F : X \mapsto \mathbb{R}^m$. Then the property '$f$ is transverse to Y' is open.

The proof of the transversality theorem is not simple, and will not be reproduced here. Further, it is fairly easy to see that under the conditions of the theorem, the property 'f is transverse to Y' is dense. Therefore, transversality is a generic property. Now, a closed plane curve γ produces an image under the Monge-Taylor map, μ_γ, which is also a closed curve. Thus μ_γ can be written as function of S^1, which is compact. The image of γ lies in V_k, which we identify with $\mathbb{R}^{k-1}$. For γ to have the properties we are interested in, μ_γ must intersect certain subsets of V_k. These subsets form smooth manifolds in V_k, which we shall denote by Y. So we have set of smooth mappings $\mu_\gamma : S^1 \mapsto \mathbb{R}^{k-1}$, which means the property 'μ_γ is transverse to Y' is generic.

As stated in a previous section, when μ_γ is transverse to the manifolds corresponding to vertices and inflexions, then it has no higher vertices or higher inflexions, and only finitely many ordinary vertices and inflexions. This applies to closed curves, but the results obtained apply to any part of the closed curve, which can then be viewed as an open curve in its own right. Even with an infinite open curve, we can still apply the result to any segment. Thus a generic open curve will have no higher vertices or inflexions, and any finite part of it will have only finitely many ordinary vertices and inflexions.

6 Affine curves

So far we only have considered properties of curves in Euclidean space. These properties are invariant under Euclidean transformations (translations, rotations, reflections and compositions thereof), which are also referred to as 'rigid motions'. For example, a plane curve is determined by its curvature up to rigid motion. In this Chapter we shall extend our ideas to curves in affine space. After a brief introduction to affine transformations, we shall extend the notions of curvature, normal vector, evolute and the distance-squared function to the affine case. Further, after introducing sextactic points (the affine version of vertices), we shall show how their existence or otherwise reveals local information about the form of the affine evolute.

6.1 Euclidean and affine transformations

A Euclidean transformation of a point can be defined as a map $f : \mathbf{x} \in \mathbb{R}^n \mapsto A\mathbf{x} + \mathbf{b}$, for some $\mathbf{b} \in \mathbb{R}^n$ and some $A \in O_n(\mathbb{R})$, the set of all real n by n matrices that have their transpose equal to their inverse, known as the orthogonal group. A Euclidean transformation on a curve or surface just transforms every point on that curve or surface. Geometrically, any Euclidean transformation is a translation; a rotation about a point; a reflection in a line; or a glide reflection (reflection in a line and a translation parallel to that line). Any composition of a Euclidean map is a Euclidean map, and thus they form a group under composition. An important feature of Euclidean maps is that they preserve distance (and hence angles and shapes).

An affine transformation is different only in that the matrix A need only be in $GL_n(\mathbb{R})$, the set of all real n by n matrices that have an inverse, known as the general linear group. Geometrically, affine maps cannot be described as simply as Euclidean maps. This is because given $n+1$ points in general position (i.e. no $d+1$ of them lie in a $(d-1)$-dimensional plane) there exists a unique n-dimensional affine map which maps these $n+1$ points to any other $n+1$ points in general position. Taking the 2D case, a Euclidean map can map any parallelogram (or ellipse) in the plane to any other parallelogram (or ellipse) *of the same shape and size*. An affine map can map any parallelogram (or ellipse) in the plane to any other parallelogram (or ellipse), regardless of the shape or size of the second parallelogram (or ellipse). In higher dimensions, the same is true for parallelepipeds and ellipsoids (including spheres).

This is the main reason why the curvature function used up to now will not suffice. Recall that the curvature can be defined via $(n+1)$-point contact with an n-dimensional sphere, and then considering the radius of that sphere. This works in the Euclidean case because a Euclidean map preserves lengths, and therefore the radius of the sphere. Thus, the curvature at a point of a curve is invariant under a Euclidean map. However, an affine map does not preserve the radius of a sphere, and so what we will now call the 'Euclidean curvature' is not invariant under an affine map. In the next section, the concept of 'affine curvature' is introduced to resolve this.

6.2 Basic notions

From this section onwards, we choose co-ordinates for the affine plane such that the area of the parallelogram spanned by two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ is $|\mathbf{a}\,\mathbf{b}| = a_2 b_2 - a_2 b_1$. Let $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^2$ be a curve in the affine plane, with $\left|\dot\beta\,\ddot\beta\right| \neq 0$ (meaning β has no inflexions), where $\dot\beta = \frac{d\beta}{dt}$. Then the 'affine arc length' between $\beta(a)$ and $\beta(b)$ is given by $s(t) = \int_a^b \left|\dot\beta\,\ddot\beta\right|^{1/3} dt$. It is possible to choose a parameter s such that $|\beta'(s)\,\beta''(s)| = 1$ (where $\beta' = \frac{d\beta}{ds}$), and then we say that β is 'parameterised by affine arc length'. This corresponds to β being unit speed in the Euclidean case.

In the Euclidean case, one of the the 2-dimensional Serret-Frenet formulae states that $\mathbf{T}' = \frac{d^2\gamma}{dt^2} = \kappa\mathbf{N}$. It is possible to show that the following similar formula applies in the affine case:

$$\beta'''(s) = -k_a(s)\beta''(s)$$

where $k_a(s) = |\beta''(s)\,\beta^{(3)}(s)|$ is the 'affine curvature' of β. The main difference with the Euclidean curvature is that the affine curvature involves the 3rd derivative of the curve, whereas the Euclidean version only involves the 2nd derivative. (The Euclidean curvature can be defined by $\kappa(s) = |\beta'(s)\,\beta''(s)|$). The vector $\beta''(s)$ is called the 'normal vector' of β, and the point $\beta(s_0) + \beta''(s)/k_a(s_0)$ is called the 'affine centre of curvature' of β at s (assuming that $k_a(s) \neq 0$. The locus of the affine centre of curvature is called the 'affine evolute' of β, and is given by $\xi_a = \beta(s) + \beta''(s)/k_a(s)$. The curve $\beta'' : I \subset \mathbb{R} \mapsto \mathbb{R}^2$ is called the 'affine normal curve' of β.

Having defined the concept of affine curvature, the Euclidean case would suggest the need for a name for the points where the curvature vanishes or has a stationary point. It turns out that $k_a'(s) = 0$ if there exists a unique conic having six-point contact with β at the point $\beta(s)$. Such points are therefore known as 'sextactic points', and correspond to vertices in the Euclidean case. In addition, we say $\beta(s)$ is a 'parabolic point' if there exists a unique parabola with five-point contact at $\beta(s)$, and these points occur iff $k_a(s) = 0$. Such points correspond to inflexions in the Euclidean case.

Before proceeding any further, there are a few generic properties that are required for the results that follow. Although they apply specifically to closed curves, they also apply to any part of a closed curve, and thus to (open) curves that are images of an interval rather than the entire real line (i.e. those of the form $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^2$). Now, let β be a closed smooth curve without inflexions. Then the following properties of β are generic:

- No conic has greater than six-point contact with β
- There are only a finite number of points on β where there the unique non-singular conic touching β at that point is a parabola
- There is no parabola with six point contact with β.

These results correspond to a Euclidean curve having no higher vertices, only isolated ordinary inflexions, and no higher inflexions respectively.

6.3 Distance-cubed function

Given a point $\mathbf{u}$ in the affine plane, and an affine curve $\beta(s) : I \mapsto \mathbb{R}^2$, then the distance-cubed function is gievn by $F : I \times \mathbb{R}^2 \mapsto \mathbb{R}$ is given by

$$F(s, \mathbf{u}) = |\beta'(s)\; \beta(s) - \mathbf{u}|$$

(β' is used to denote $\frac{d\beta}{ds}$). This function performs a similar role to the Euclidean distance-squared function, in that it can be used to reveal interesting geometric information about the affine evolute. The most obvious difference with Euclidean case (other than the name) is the presence of a term involving the derivative of β. We note that $|\mathbf{a}\;\mathbf{b}|' = |\mathbf{a}'\;\mathbf{b}| + |\mathbf{a}\;\mathbf{b}'|$; $|\beta'''\;\beta''| = |k_a\beta''\;\beta''| = 0$ and $|\beta''\;\beta'| = 1$. Differentiating $F(s, \mathbf{u})$ with respect to s, and denoting $F(s, \mathbf{u})$ by $f_u(s)$ produces the following:

$$\begin{aligned} f_u'(s) &= |\beta''(s)\; \beta(s) - \mathbf{u}| \\ f_u''(s)a &= \left|\beta^{(3)}(s)\; \beta(s) - \mathbf{u}\right| - 1 \end{aligned}$$

$$
\begin{aligned}
f_u^{(3)}(s)a &= \left|\beta^{(4)}(s)\,\beta(s)-\mathbf{u}\right| \\
f_u^{(4)}(s)a &= \left|\beta^{(5)}(s)\,\beta(s)-\mathbf{u}\right| + \left|\beta^{(4)}(s)\,\beta'(s)-\mathbf{u}\right|
\end{aligned}
$$

From this, and using similar techniques to those used in section 4.3.1, we obtain the following results:

$$
\begin{aligned}
f'_u(s)=0 &\quad \text{iff} \quad \exists\lambda\in\mathbb{R} \text{ such that } \beta(s)-\mathbf{u}=\lambda\beta''(s) \\
f'_u(s)=f''_x(s)=0 &\quad \text{iff} \quad k_a(s)\neq 0 \text{ and } \mathbf{u}=\beta(s_0)+\frac{\beta''(s)}{k_a(s)} \\
f'_u(s)=f''_x(s)=f^{(3)}_x(s)=0 &\quad \text{iff} \quad k_a(s)\neq 0\ ,\ k'_a(s)=0 \\
&\qquad\qquad \text{and } \mathbf{u}=\beta(s)+\frac{\beta''(s)}{k_a(s)}
\end{aligned}
$$

If we recall that $\mathbf{N}=\beta''$, then these results are effectively identical to those obtained in section 4.3.1, which relate the the zero points of the derivatives of the Euclidean distance-squared function with information about the Euclidean curvature.

6.4 Sextactic points and the affine evolute

The aim of this section is extend the results from sections 4.4 and 5.3 to form similar results for affine curves. The first result relates the local structure of the affine evolute with the existence of sextactic points, and is taken from the paper by Izumiya and Sano [7]. With a Euclidean curve, the presence of a vertex on a curve indicates that the evolute is locally diffeomorphic to a cusp at the corresponding point; otherwise the evolute is smooth. The result extends naturally to an affine curve as follows:

Theorem Given an affine plane curve $\beta : I\subset\mathbb{R}\mapsto\mathbb{R}^2$, its affine evolute $\xi_a(s)$ is locally diffeomorphic to a cusp at a point $\xi_a(s_0)$ if $k'_a(s_0)=0$ (i.e. $\beta(s_0)$ is a sextactic point). Further, $\xi_a(s)$ is locally diffeomorphic to a line at a point $\xi_a(s_0)$ if $k'_a(s_0)\neq 0$ (i.e. $\beta(s_0)$ is not a sextactic point).

The proof of this theorem very closely follows the Euclidean version of this result, which is given in section 4.4. As the distance-cubed function depends on two parameters, section 4.4 tells us that the bifurcation set must be diffeomorphic to a line or a cusp. It can be shown that the bifurcation set of the distance-cubed function is exactly the affine evolute, and the theorem follows simply from this.

In section 5.3, we saw that a generic closed Euclidean plane curve has only a finite number of vertices - points where the curvature has a stationary point. This and the theorem above would suggest that the following proposition is true:

Proposition A generic closed affine plane curve has only a finite number of sextactic points.

The first step is to define the notation to be used. Given a closed affine plane curve $\beta : S^1 \mapsto \mathbb{R}^2$, it can be expressed as a function of Euclidean arc length t or affine arc length s. Recall from section 5.1 that we can write a curve in the Monge normal form $\eta = f_t(\zeta)$, where the co-ordinate axes ζ and η are parallel to the unit tangent and normal vectors at $\gamma(t)$ and $(\zeta, \eta) = (0, 0)$ at the point $\gamma(t)$. Note that ζ and η are functions of t. We shall denote the Euclidean curvature by κ, $\frac{d\kappa}{d\zeta}$ by κ', $\frac{d\kappa}{dt}$ by $\dot{\kappa}$, and affine curvature by k_a.

From [6], we find that the point $\gamma(t)$ is a sextactic point iff

$$36\kappa(t)^4\dot{\kappa}(t) + 40\dot{\kappa}(t)^3 - 45\kappa(t)\dot{\kappa}(t)\ddot{\kappa}(t) + 9\kappa(t)^2\frac{d^3\kappa}{dt^3}(t) = 0$$

The aim is to show that this condition can be re-expressed as a single polynomial in co-ordinates from the Monge-Taylor space.

$$\begin{aligned}
\dot{\kappa} = \frac{d\kappa}{dt} &= \frac{d\kappa}{d\zeta}\frac{d\zeta}{dt} \text{ (chain rule)} \\
&= \frac{d\kappa}{d\zeta}\left|\frac{d\mathbf{T}}{dt}\right| \text{ (tangent parallel to } \zeta\text{-axis)} \\
&= \kappa'\left|\dot{\mathbf{T}}\right| \\
&= \kappa'\left|\kappa\mathbf{N}\right| \text{ (Serret-Frenet)} \\
&= \kappa'\kappa \\
\text{Similarly, } \ddot{\kappa} &= \kappa''\kappa + \kappa'^2\kappa \\
\text{and } \frac{d^3\kappa}{dt^3} &= \kappa'''\kappa^2 + \kappa''\kappa'\kappa + 2\kappa''\kappa^2 + (\kappa')^2\kappa
\end{aligned}$$

Further, from [2], we find that κ and its derivatives with respect to ζ can be given as functions of co-ordinates in the Monge-Taylor space. Specifically:

$$\begin{aligned}\kappa(t) &= 2a_2(t)\\ \kappa'(t) &= 6a_3(t)\\ \kappa''(t) &= 24(a_4(t) - a_2(t)^3)\end{aligned}$$

Further, it can be shown that $\kappa'''(t)$ can be given as a polynomial in a_i. Using this, we can produce a polynomial $f(a_i(t))$ such that $\gamma(t)$ is a sextactic point iff $f(a_i(t)) = 0$. We then consider the Monge-Taylor map of degree at least equal to $n + 1$, where n is the maximum value of i in $f(a_i(t))$. The Monge-Taylor space then has degree at least equal to n. The polynomial $f(a_i(t)) = 0$ defines a subspace of codimension 1 within the Monge-Taylor space, and γ has a sextactic point if and only if the curve μ_γ (the image of the curve under the Monge-Taylor) intersects this subspace.

The curve μ_γ is of dimension 1, and if it intersects a subspace of codimension 1 transversally, then the intersection is an isolated point. We know from section 5.3 that transversality is a generic property of a closed curve, and thus a generic affine plane curve only has a finite number of sextactic points. This completes the proof.

6.5 Invariance

Invariance is an important property in geometry, as properties that remain invariant are under certain changes are in a sense more fundamental properties of the object in question. In this section, we will see how affine curvature is not invariant under a general affine transformation, but only under a subset, known as equi-affine transformations. However, it turns out that zero points of the affine curvature and its derivatives remain as such under a general affine transformation.

Recall from section 6.2 that the affine curvature of a unit speed affine plane curve $\beta(s) = (\beta_1(s), \beta_2(s)) : I \subset \mathbb{R} \mapsto \mathbb{R}^2$ is given by $k_a(s) = |\beta''(s), \beta'''(s)| = \beta_2''\beta_1''' - \beta_1''\beta_2'''$. We shall take a general affine transformation, and find the conditions for A and $\mathbf{B}$ that ensure k_a is unchanged.

Let $\tilde{\beta} = A\beta + \mathbf{b}$, with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(\mathbb{R}) \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}_2$$

We write $\tilde{k}_a$ for the curvature of $\tilde{\beta}$. Then we have that

$$\tilde{\beta}'' = \begin{pmatrix} a_{11}\beta_1'' + a_{12}\beta_2'' \\ a_{21}\beta_1'' + a_{22}\beta_2'' \end{pmatrix} \text{ and } \tilde{\beta}''' = \begin{pmatrix} a_{11}\beta_1''' + a_{12}\beta_2''' \\ a_{21}\beta_1''' + a_{22}\beta_2''' \end{pmatrix}$$

The affine curvature of $\tilde{\beta}$ is then

$$\begin{aligned} \tilde{k}_a &= (a_{21}\beta_1'' + a_{22}\beta_2'')(a_{11}\beta_1''' + a_{12}\beta_2''') \\ &\quad -(a_{11}\beta_1'' + a_{12}\beta_2'')(a_{21}\beta_1''' + a_{22}\beta_2''') \\ &= a_{11}a_{21}(\beta_1''\beta_1''' - \beta_1''\beta_1''') + a_{12}a_{21}(\beta_1''\beta_2''' - \beta_2''\beta_1''') \\ &\quad +a_{11}a_{22}(\beta_1'''\beta_2'' - \beta_1''\beta_2''') + a_{12}a_{22}(\beta_2''\beta_2'' - \beta_2''\beta_2'') \\ &= (a_{12}a_{21} - a_{11}a_{22})(\beta_1''\beta_2''' - \beta_2''\beta_1''') \\ \text{therefore } \tilde{k}_a &= |A|\, k_a \end{aligned}$$

In the special case $k_a = 0$, there are no restrictions on either A or $\mathbf{b}$, as in this case we have $\tilde{k}_a = k_a$. In the general case, we have no restriction on $\mathbf{b}$, but must have $|A| = 1$ in order to satisfy $\tilde{k}_a = k_a$. Therefore, transformations that preserve affine curvature are of the form $f : \mathbf{x} \in \mathbb{R}^2 \mapsto A\mathbf{x} + \mathbf{b}$, for some $\mathbf{b} \in \mathbb{R}^2$ and some $A \in SL_2(\mathbb{R})$, the set of all real 2 by 2 matrices with determinant one, known as the special linear group group. An important feature of equi-affine transformations is that they preserve area.

A similar result holds for $k_a'(s)$. Its value it preserved in general by equi-affine transformations, and points where $k_a' = 0$ being preserved under general affine transformations. Thus the property that a point $\beta(s_0)$ is a sextactic point is invariant under a general affine transformation. With curves in n dimensions, the results generalise in the natural way: affine curvature is, in general, only preserved under the action of members of $SL_n(\mathbb{R})$; points where the curvature or its derivatives is zero remain as such under a general affine transformation.

6.6 Affine space curves

In this section, we extend the ideas relating to affine plane curves to three dimensions. We start by choosing co-ordinates for the affine space such that the volume of the parallelepiped spanned by the vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ is given by the determinant

$$|\mathbf{a}\,\mathbf{b}\,\mathbf{c}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^3$ be an affine space curve, with $\left|\dot{\beta}\;\ddot{\beta}\;\frac{d^3\beta}{d\beta^3}\right| \neq 0$, where $\dot{\beta} = \frac{d\beta}{dt}$. Then the 'affine arc length' of β between two points $\beta(a)$ and $\beta(b)$ is given by $s(t) = \int_a^b \left|\dot{\beta}\;\ddot{\beta}\;\frac{d\beta}{dt}\right|^{1/6} dt$. It is possible to choose a parameter s such that $|\beta'(s)\,\beta''(s)\,\beta'''(s)| = 1$ (where $\beta' = \frac{d\beta}{ds}$), and then we say that β is 'parameterised by affine arc length'.

The affine curvature of an affine space curve $\beta(s)$ is given by $k_a(s) = \left|\beta'(s)\,\beta'''(s)\,\beta^{(4)}(s)\right|$. Then one can define the 'affine tangent' vector $\mathbf{T}(s) = \beta'(s)$; the 'principle affine normal' vector $\mathbf{N}(s) = \beta''(s)$; the 'intrinsic affine binormal' vector $\mathbf{B}(s) = k_a(s)\beta'(s) + \beta'''(s)$; and the 'intrinsic affine torsion' $\sigma_a(s) = -|\beta''(s)\beta'''(s)\,\beta^{(4)}(s)| - k_a'(s)$.

(A brief aside into notation for the affine torsion: this is drawn from the paper by Izumiya & Sano [8], who initially define a different affine torsion $\tau_a(s) = -|\beta''(s)\beta'''(s)\,\beta^{(4)}(s)|$. However, this and the initial definition of the binormal vector used turn out to be less useful than $\sigma_a(s)$. Although this work omits the intermediate stages and uses the $\sigma_a(s)$, it was thought best to keep notation as consistent as possible with [8]).

The following set of equations, similar to the Serret-Frenet equations, link the tangent, normal and binormal vectors and their derivatives:

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa_a(s) & 0 & 1 \\ -\sigma_a(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}$$

6.6.1 Distance-sixth function and its singularities

The three dimensional analogue of the distance cubed function is called the 'distance-(6th-powered) function', hereafter called the 'distance-sixth function'. The definition extends naturally from two dimensions to three. The distance-sixth function $F : I \times \mathbb{R}^3 \mapsto \mathbb{R}$ is given by:

$$F(s, \mathbf{u}) = |\beta'(s)\,\beta''(s)\,(\beta(s) - \mathbf{u})|$$

We denote $F(s, \mathbf{u})$ by $f_u(s)$ and define $M_{ij}(s) = |\beta^{(i)}(s)\, \beta^{(j)}(s)\, (\beta(s) - \mathbf{u})|$. Note that $|\mathbf{a}\,\mathbf{b}\,\mathbf{c}|' = |\mathbf{a}'\,\mathbf{b}\,\mathbf{c}| + |\mathbf{a}\,\mathbf{b}'\,\mathbf{c}| + |\mathbf{a}\,\mathbf{b}\,\mathbf{c}'|$. Then differentiating $F(s, \mathbf{u})$ with respect to s produces the following:

$$\begin{aligned} f_u' &= M_{13} \\ f_u'' &= M_{23} - k_a M_{12} \\ f_u^{(3)} &= (\sigma_a + k_a' - k_a) M_{12} \\ f_u^{(4)} &= (\sigma_a'' + k_a''' - k_a'' + k_a^2) M_{12} + (\sigma_a - k_a') M_{13} - k_a M_{23} \end{aligned}$$

From this, and using similar techniques to those used in section 4.3.1, we obtain the following results:

$$\begin{aligned} f_u'(s) = 0 \text{ iff } \beta(s) - \mathbf{u} &= \lambda \beta'(s_0) + \mu \beta'''(s_0) \;\; \text{for some } \lambda, \mu \in \mathbb{R} \\ f_u'(s) = f_x''(s) = 0 \;\; &\text{iff} \;\; \lambda = \tilde{\lambda} k_a(s) \text{ for some } \tilde{\lambda} \in \mathbb{R} \text{ ; and } \mu = -1 \\ &\qquad \text{i.e. } \beta(s) - \mathbf{u} = \tilde{\lambda} k_a(s) \beta'(s_0) - \beta'''(s) \\ f_u'(s) = \ldots = f_x^{(3)}(s) = 0 \;\; &\text{iff} \;\; \sigma_a(s) \neq 0 \text{ , } \tilde{\lambda} = 1/\sigma_a(s) \\ &\qquad \text{i.e. } \beta(s) - \mathbf{u} = \frac{k_a(s)}{\sigma_a(s)} \beta'(s) - \beta'''(s) \\ f_u'(s) = \ldots = f_x^{(4)}(s) = 0 \;\; &\text{iff} \;\; \sigma_a(s) \neq 0 \text{ , } \sigma_a'(s) = 0 \\ &\qquad \text{and } \beta(s) - \mathbf{u} = \frac{k_a(s)}{\sigma_a(s)} \beta'(s_0) - \beta'''(s_0) \end{aligned}$$

It can be shown that with subsequent singularities of the distances-sixth function result in successive derivatives of σ_a vanishing. Comparing the results with those for the distance-squared function and Euclidean curves (subsection 4.3.2), we see more differences than we did for in the two dimensional case. However, there is still some similarities. For both the Euclidean distance-squared function and the affine distance-sixth function, an A_1 singularity means that $\mathbf{u}$ is restricted to a plane, while an A_2 singularity means that $\mathbf{u}$ is restricted to a line. Assuming that the torsion is not zero, an A_3 singularity means that $\mathbf{u}$ is restricted to a point.

6.6.2 Affine rectifying Gaussian surface

The 'affine rectifying Gaussian surface' is defined to be the set $GS[\beta(s)] = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} = \lambda\beta'(s) + \beta^{(3)}(s) \text{ for some } \lambda \in \mathbb{R}\}$. It performs a similar role in three dimensions as the evolute does in two, in that its local structure is related to the type of singularity of the distance-sixth function. The following theorem comes from [8]:

Theorem Consider an affine space curve $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^3$ and its affine rectifying Gaussian surface $GS[\beta]$. Then at a point $\beta(s_0)$ on the curve, $GS[\beta(s_0)]$ is locally diffeomorphic to:

- a plane in if the distance-sixth function has an A_3 singularity at $\beta(s_0)$
- a cuspoidal edge if the distance-sixth function has an A_4 singularity at $\beta(s_0)$
- the swallowtail surface if the distance-sixth function has an A_3 singularity at $\beta(s_0)$

6.7 Affine curves in higher dimensions

In this section, we extend the ideas of affine plane and space curves to an arbitrary number number of dimensions. We start by setting co-ordinates such that given a set of n vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ in $\mathbb{R}^n$, then the volume of the parallelepiped spanned by these vectors is equal to the determinant whose ith column is $\mathbf{v_i}$. We denote this determinant by $|\mathbf{v_1}\ \mathbf{v_2}\ \ldots\ \mathbf{v_n}|$.

Let $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^3$ be a curve in the affine plane, with $\left|\dot{\beta}\ \ddot{\beta}\ \ldots\ \frac{d^n\beta}{d\beta^n}\right| \neq 0$, where $\dot{\beta} = \frac{d\beta}{dt}$. Then the 'affine arc length' of β, $s(t)$, between two points $\beta(a)$ and $\beta(b)$ is given by

$$s(t) = \int_a^b \left| \frac{d\beta}{d\beta}\ \frac{d^n\beta}{d\beta^2}\ \ldots\ \frac{d^n\beta}{d\beta^n} \right|^{\frac{2}{n(n+1}} dt$$

It is possible to choose a parameter s such that $\left|\frac{d\beta}{ds}\ \frac{d^2\beta}{ds^2}\ \ldots\ \frac{d^n\beta}{ds^n}(s)\right| = 1$, and then we say that β is 'parameterised by affine arc length'. We shall assume that all our curves are parameterised by affine arc length.

For an affine curve in n dimensions, it is always possible to find functions of s, denoted $\nu_1, \nu_2, \ldots, \nu_{n-1}$, such that

$$\nu_1 \frac{d\beta}{ds} + \nu_2 \frac{d^2\beta}{ds^2} \ldots + \nu_{n-1} \frac{d^{(n-1)}\beta}{ds^{(n-1)}} + \frac{d^n\beta}{ds^n} = 0$$

Having found these, we define the 'generalised affine curvatures', denoted by $\phi_1, \ldots \phi_{n-1}$ for an affine curve in n dimensions by

$$\phi_i = \sum_{j=1}^{j=i} \left[(-1)^{i+j} \frac{(n-j-1)!}{(i-j)!(n-i-1)!} \nu_{n-j}^{(i-j)} \right]$$

$\nu_{n-j}^{(i-j)}$ denotes the $(i-j)$th derivative of ν_{n-j} with respect to s. The first few ϕ_i are given as follows:

$$\begin{aligned}
\phi_1 &= \nu_{n-1} \\
\phi_2 &= -(n-2)\nu'_{n-1} + \nu_{n-2} \\
\phi_3 &= \frac{1}{2}(n-2)(n-3)\nu''_{n-1} - (n-3)\nu'_{n-2} + \nu_{n-3} \\
\phi_4 &= -\frac{1}{6}(n-2)(n-3)(n-4)\nu'''_{n-1} + \frac{1}{2}(n-3)(n-4)\nu''_{n-2} \\
&\quad -(n-4)\nu'_{n-3} + \nu_{n-4}
\end{aligned}$$

The generalised affine curvatures have similar properties to the generalised Euclidean curvatures. For example, if $\phi_{i-1} \neq 0$ and $\phi_i = \phi_{i+1} = \ldots = \phi_n = 0$, then β is restricted to an $(n-i+1)$-dimensional linear subspace of $\mathbb{R}^n$. Further, denoting the tangent vector by $\mathbf{A_1}$, the generalised affine curvatures can be used to define a set of frame vectors via the following Serret-Frenet-like set of formulae:

$$\begin{pmatrix} \mathbf{A_1}' \\ \mathbf{A_2}' \\ \mathbf{A_3}' \\ \vdots \\ \mathbf{A_n}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ -\phi_1 & 0 & 1 & \ddots & \vdots \\ -\phi_2 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ -\phi_{n-1} & 0 & \ldots & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{A_1} \\ \mathbf{A_2} \\ \mathbf{A_3} \\ \vdots \\ \mathbf{A_n} \end{pmatrix}$$

6.7.1 Distance-power function and its singularities

By now it should come as no surprise that the next step is to define an extension of the affine plane's distance-cubed function to n dimensions. On the basis that we had the 'distance-(6th-power) function' in three dimensions, technically the n-dimensional version should be called the 'distance-$(2/n(n+1)$th-power) function'. However, for reasons of brevity, we shall just refer to it as the 'distance-power function'. Given a smooth affine curve $\beta : I \subset \mathbb{R} \mapsto \mathbb{R}^n$ in n dimensions, the distance power function $F(s, \mathbf{u}) : I \subset \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}$ is defined to be:

$$F(s, \mathbf{u}) = \left| \beta'(s)\, \beta''(s)\, \ldots\, \beta^{(n-1)}(s)(\beta(s) - \mathbf{u}) \right|$$

Naturally, we are interested in the geometric consequences of the distance-power function having an A_k singularity. The following theorem from [5] tell us the required information:

Theorem Given an n-dimensional curve $\beta(s)$, the distance-power function $F(s, \mathbf{u})$ on that curve has an A_k singularity for $0 \leq k \leq n-1$ iff for some specific $\lambda_i \in \mathbb{R}$

$$\mathbf{u} = \beta + \lambda_n \mathbf{A}_n + \sum_{i=1}^{n-k-1} \lambda_i \mathbf{A}_i \text{ with } \lambda_{n-k-1} \neq 0$$

Given an n-dimensional curve $\beta(s)$, the distance-power function $F(s, \mathbf{u})$ on that curve has an A_n singularity for iff $\phi_{n-1} \neq 0$, $\phi'_{n-1} \neq 0$, and

$$\mathbf{u} = \beta + \frac{\mathbf{A}_n}{\phi_{n-1}}$$

Given an n-dimensional curve $\beta(s)$, the distance-power function $F(s, \mathbf{u})$ on that curve has an A_{n+1} singularity for iff $\phi_{n-1} \neq 0$, $\phi'_{n-1} = 0$, $\phi''_{n-1} \neq 0$, and

$$\mathbf{u} = \beta + \frac{\mathbf{A}_n}{\phi_{n-1}}$$

With $0 < k < n$, an A_k singularity means that $\mathbf{u}$ is confined to some linear subspace of codimension k, given by $\{\mathbf{v} \in \mathbb{R}^n | \mathbf{v} = \lambda_n \mathbf{A}_n + \sum_{i=1}^{n-k-1} (\lambda_i \mathbf{A}_i) + \text{span}\{\mathbf{A}_{n-k}\ ,\ \mathbf{A}_{n-k+1}\ ,\ \ldots\ ,\ \mathbf{A}_{n-1}\}\}$, with λ_i fixed. With $k \geq n$, an A_k singularity means that $\mathbf{u}$ is a fixed point, with additional restrictions relating to the generalised curvature and its derivatives. These results agree with those obtained in section 4.3.3 for the (Euclidean) distance-squared function in n dimensions and its singularities. In that section, we also saw a specific way of establishing the value of λ_i, and similar techniques could probably be used for the affine case.

7 Further work and conclusion

This final chapter summarises the results obtained from this work, and looks at possible extensions of the ideas into areas that have already been covered, and possible new areas.

7.1 Height functions

Of existing areas, the most major topic not covered in this work relates to the height function, which was defined in section 2.4. We have seen how the distance-squared function and its affine variants can be used to pick out vertices (where the derivative of the relevant curvature is zero), and to reveal the local structure of the evolute of a curve. Similarly, the height function and its affine variants can be used to pick out points of inflexion (where the relevant curvature is zero), and to reveal the local structure of the 'dual curve'. (Given a curve not passing through the origin, the locus of the foot of the perpendicular to the tangent line passing through the origin is called the 'pedal' of that curve, and is an example of a dual curve).

In the affine case, points of inflexion are replaced by parabolic points, but the results translate from the Euclidean case in a similar way to those for the distance-squared function. The Euclidean case is only covered up to three dimensions in [2], where as the affine case for two, three and n dimensions is covered in [7], [8] and [5] respectively. An obvious new area is extending the Euclidean results into n dimensions, possibly using similar techniques to those used in section 4.3.3.

7.2 Plane and hyperplane height functions

Given the distance functions are concerned with the distance of a curve from a point, and the height functions are concerned with the distance of a curve from a line, the natural extension of these functions to consider the distance from a plane. This function will be called the 'plane height function', and the traditional height function will hereafter be called the 'line height function' (calling it the 'plain height function' would only lead to confusion). Clearly the plane height function is only defined for curves in three or more dimensions.

Intuition suggest that the plane height function can be used to pick out points where the torsion is zero, which will be referred to as 'planar points'.

(Note that if the curvature is zero, then the torsion is zero also). The reason for this is as follows. At a point of inflexion, the curvature is zero, and so the contact of the curve with a the tangent line will be of higher degree than normal, which gives a rough idea of why the line height function will have a singularity at a point of inflexion. Similarly, at a planar point, the torsion is zero, and so the contact of the curve with a the tangent plane (the plane spanned by the tangent and normal vectors) will be of higher degree than normal - hence the plane height function should have a singularity at a planar point.

The distance function and line height functions also tell us information about the local structure of the evolute and dual curves respectively, it is worth asking if the plane height function can be used in a similar way. The notion of a dual curve can be extended as follows. Given a curve not passing through the origin, the 'trial curve' is defined as the locus of the foot of the perpendicular to the tangent plane passing through the origin. ('Trial' coming from 'tri' + 'al'). It is expected that the plane height function can be used to reveal information about the local structure of the trial curve.

After looking at distance from a plane, the next step is consider the 'hyperplane height function', which looks at the distance of an n-dimensional curve from an z-dimensional hyperplane ($0 < z < n$). A similar line of reasoning to that used for plane height function suggests that the plane height function can be used to pick out points where χ_z is zero, which will be referred to as 'hyperplanar points'. Similarly, one can define the $(z-1)$-al curve as s the locus of the foot of the perpendicular to the tangent z-dimensional hyperplane passing through the origin. The tangent z-dimensional hyperplane is that spanned by the frame vectors $\mathbf{E}_1, \ldots, \mathbf{E}_z$. (For specific values of z, these curves can be referred to as the dual curve, trial curve, quadral curve, quintal curve, etc.).

Naturally, the plane and hyperplane distance functions could be generalised into affine versions to see if the Euclidean results and affine results are similar. It would appear that neither the plane height nor the hyperplane height function have been studied before, and so investigations into their properties would presumably produce original results.

7.3 Link between Euclidean and affine cases

For both the height and distance functions, is would be useful to see how far the similarities in results between the affine and Euclidean case goes. With the distance function, it appears that any result about Euclidean curves can be translated into a result for affine curves by replacing the words 'curvature', 'vertex' and 'distance-squared function' withe the words 'affine curvature', 'sextactic point' and 'distance-power function' respectively. For example, one could see if the four vertex theorem (which states every closed plane curve at least four vertices) implies a 'four sextactic point' theorem (i.e. every closed plane curve has at least four sextactic points). As the proof of the former is long and complicated, it would useful to prove the link between the results instead.

One possible way of doing this would be as follows. Given any plane curve $\beta(s)$ the aim would be to find an equi-affine map $A(\beta)$, given by $A : \mathbf{x} \mapsto A\mathbf{x}$ for some $A \in SL_2(\mathbb{R})$, which produces a new curve $\tilde{\beta}(s) = A\beta(s)$ such that *the Euclidean curvature of $\tilde{\beta}(s)$ is equal to the affine curvature of* $\beta(s)$ for all points on the curve, that is,

$$\kappa\left[\tilde{\beta}(s)\right] = k_a\left[\beta(s)\right] \text{ for all } s$$

As affine curvature is preserved under an equi-affine map, then 'sextactic point at $\beta(s_0)$' $\Leftrightarrow$ 'sextactic point at $\tilde{\beta}(s_0)$' $\Leftrightarrow k_a'[\tilde{\beta}(s_0)] = 0 \Leftrightarrow \kappa'[\tilde{\beta}(s_0)] = 0$ $\Leftrightarrow$ '$\tilde{\beta}$ has a vertex'. (Note that if two smooth functions are equal at all points, then their derivatives are equal at all points). A similar statement can be made relating parabolic points and points of inflexion.

If such a map can be shown to exist, then this would provide the means to translate results for Euclidean curves directly into results for affine curves. It may be that such an equi-affine map does not exist, in which case, it would be sufficient to use any map which preserves affine curvature instead.

For more than two dimensions, the situation becomes more complicated, as a map would need to be found that preserves the generalised affine curvatures, and the generalised Euclidean and affine curvatures of the image of a curve under the map would have be equal. However, if this could be done, then the results for the plane and hyperplane height functions could also be directly translated into affine versions.

7.4 Miscellaneous

There a few more minor areas that have potential for further investigation. The first relates to the the statement given at the end of section 4.4.2 - specifically that the evolute being locally diffeomorphic to something other than a straight line indicates the presence of a vertex or a point where $\chi_2 = 0$. The question remains as whether the converse is true: will the evolute of a space or higher dimensional curve have a cusp at points corresponding to vertices?

The theorem given in section 6.7.1 states that for an n-dimensional curve, an A_k singularity ($0 < k < n$) of the distance-squared function $F(s, \mathbf{u})$ means that u is confined to some linear subspace of codimension k. This subspace is given in terms of k co-efficients denoted by λ_i. It could be interesting to derive more explicit expressions for λ_i in a similar way to that used in section 4.3.3. The affine and Euclidean cases could then be compared for similarities and differences.

One important concept is that all curves in this work have inhabited $\mathbb{R}^n$ with the standard metric, and have had no restrictions about where the curve can be. However, one can also look at curves that are restricted to a given r-dimensional hypersurface in $\mathbb{R}^{n>r}$. This is equivalent to considering curves in $\mathbb{R}^r$ that use a non-standard metric. The hypersurface can be given as expressed in the form $F : \mathbf{u} \in \mathbb{R}^r \mapsto \mathbb{R}^n$, with $F(\mathbf{u}) = (f_1(\mathbf{u}) \, , \, f_2(\mathbf{u}) \, , \, \dots \, , \, f_n(\mathbf{u}))$; and the curve can be given as expressed in the form $\alpha : t \in \mathbb{R} \mapsto \mathbb{R}^r$, with $\alpha(t) = (\alpha_1(t) \, , \, \alpha_2(t) \, , \, \dots \, , \, \alpha_r(t))$. The composition of these two functions, $F \circ \alpha : t \in \mathbb{R} \mapsto \mathbb{R}^n$ will be a curve on the hypersurface. Then one can aim to relate properties of the curve on the hypersurface in $\mathbb{R}^n$ by combining its properties as a free curve in $\mathbb{R}^r$ with those of the hypersurface itself in $\mathbb{R}^n$.

Chapter 5 looked in detail at the generic properties of Euclidean plane curves, and Chapter 6 mentioned some generic properties of affine plane curves. However, no mention was made of generic properties of curves residing in three or more dimensions. Given the success of the Monge-Taylor map in two dimensions, one approach would be to try and extend this map into n dimensions. Any point curve in n dimensions will locally look like a graph. One can then use the frame vectors to define local co-ordinate axes, and hence produce an n-dimensional Monge normal form. The leads to its $(n-1)$-dimensional Taylor expansion, truncated at degree k, to produce a polynomial of degree k in $n-1$ variables. The co-efficients of this polynomial are a function of the parameter of the curve. The n-dimensional

Monge-Taylor map take the curve to the vector space of the co-efficients of this polynomial. Possible properties of the curve (such as the existence of higher vertices or similar) could potentially be translated into conditions in the Monge-Taylor space, and transversality used to establish whether or not such properties are generic.

7.5 Conclusion

This work has aimed to give a complete overview of the relationship between a curve in any number of dimensions and the distance-squared function and its affine variants. For Euclidean curves, this was achieved, with the implications of a the distance squared function n dimensions having an A_k singularity given explicitly. The general results for n dimensions were new. For the affine variants, similar results were produced, although the implications were of descriptive nature rather than an exact formulation. As part of this, versal unfoldings were used to computer the local structure of the bifurcation set of any r-parameter unfolding possessing an A_k singularity. Further, the generic properties of a Euclidean plane curve were established, and a new generic property of affine curves was demonstrated.

Having investigated in detail the results for the distance functions, a brief overview of known results for the (line) height function was given, and the expected results for the plane and hyperplane functions. In addition, the generic properties of both Euclidean and affine plane curves were studied, and in the latter case, a novel result was proved. A brief overview of a possible way to extend the techniques used to more than two dimensions was discussed.

8 Glossary

- α - Euclidean space curve
- β - Affine curve
- γ - Euclidean plane curve
- ζ - Monge normal form co-ordinate axis, parallel to the tangent of the curve
- η - Monge normal form co-ordinate axis, parallel to the normal of the curve
- θ - Angle between tangent line to a curve and the x-axis
- κ - Euclidean curvature ($\kappa = \chi_1$)
- λ_i - See section 4.3.3 (λ is also used to indicate some real number).
- μ_γ, μ - Monge-Taylor map of a curve γ
- ν_i - See section 6.7
- ξ - (Euclidean) evolute of a curve
- ξ_a - Affine evolute of a curve
- σ_a - Intrinsic affine torsion ($\sigma_a = \phi_2$)
- τ - Euclidean torsion ($\tau = \chi_2$)
- τ_a - A variant of affine torsion used in [8] and only mentioned in this work in an aside.
- ϕ_i - Generalised affine curvatures
- χ_i - Generalised Euclidean curvatures
- ψ_r^i - Co-efficient of $\mathbf{E}_i$ in the rth derivative of $D_{\mathbf{u}}(t)$
- A_k - Singularity of degree k.
- $\mathbf{A}_i$ - Affine frame vector

- $\mathbf{B}$ - Binormal vector ($\mathbf{B} = \mathbf{E}_3$ or $\mathbf{A}_3$)
- $D_{\mathbf{u}}(t)$ - Distance-squared function, measured from a point with position vector $\mathbf{u}$
- $\mathbf{E}_i$ - Euclidean frame vector
- $H_{\mathbf{u}}$ - Height function
- $GL_n(\mathbb{R})$ - General linear group (Real n by n matrices with non-zero determinant)
- k_a - Affine curvature ($k_a = \phi_1$)
- $\mathbf{N}$ - Normal vector ($\mathbf{N} = \mathbf{E}_2$ or $\mathbf{A}_2$)
- $O_n(\mathbb{R})$ - Orthogonal group (Real n by n matrices with transpose equal to inverse)
- $\mathbb{R}$ - Real numbers
- $SL_n(\mathbb{R})$ - Special linear group (Real n by n matrices with determinant equal to one)
- $\mathbf{T}$ - Tangent vector ($\mathbf{T} = \mathbf{E}_1$ or $\mathbf{A}_1$)

References

[1] V. I. Arnold: *Singularities of Caustics and Wave Fronts*
Kluwer Academic Publishers, The Netherlands. 1990

[2] J.W. Bruce & P.J. Giblin: *Curves and Singularities*, 2nd edition.
Cambridge Univeristy Press. 1992

[3] Th. Brücker & L. Lander: *Differentiable Germs and Catastrophes*, 2nd ed.
London Mathematical Society Lecture Notes 17, Cambridge University Press

[4] Élie Cartan: *Les systèmes différentiels extérieurs and leurs applications géométiques*
Hermann, Paris. 1971

[5] Declan Davis: *Generic affine differential geometry in* $\mathbb{R}^n$.
To appear in *Proceedings A of the Edinburgh Mathematical Society*. Available at
`www.liv.ac.uk/~pjgiblin/papers/davis.pdf` (accessed 26 March 2006)

[6] D. L. Fidal: *The existence of sextactic points*
Mathematical Proceedings of the Cambridge Philosophical Society (1984), vol 96, p433-436.

[7] Shyucihi Izumiya and Takasi Sano: *Generic affine differential geometry of plane curves*
Proceedings A of the Edinburgh Mathematical Society (1998), vol 41, p315-324

[8] Shyucihi Izumiya and Takasi Sano: *Generic affine differential geometry of space Curves*
Proceedings of the Royal Society of Edinburgh (1998), vol 128A, p301-314

[9] John McCleary: *Geometry from a Differential Viewpoint*
Cambridge Univeristy Press. 1994

[10] Wikipedia, The Free Encyclopedia: *Differential geometry of curves* (13:49 25th February 2006) (accessed 7 March 2006)
`http://en.wikipedia.org/w/index.php?title=Differential_geometry_of_curves&oldid=41166510`

[11] Mark Hatsell: *Autograph version 3.01* (computer programme). Eastmond Publishing Ltd., 2004

[12] Microsoft: *Excel 2000 (version 9.0.2720* (computer programme). Microsoft Corp., 1999

[13] Microsoft: *Paint Brush version 5.0* (computer programme). Microsoft Corp., 1999

[14] Jasc: *Paint Shop Pro version 9.0* (computer programme). Corel / Jasc Software., 2003